Theoretical Computer Science 294 (2003) 269–305
Theoretical Computer Science
www.elsevier.com/locate/tcs

Exhausting strategies, joker games and full completeness for IMLL with Unit

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Abstract

We present a game description of free symmetric monoidal closed categories, which can also be viewed as a fully complete model for Intuitionistic multiplicative linear logic with the tensor unit. We model the unit by a distinguished one-move game called *Joker*. Special rules apply to the joker move. Proofs are modelled by what we call *conditionally exhausting* strategies, which are deterministic and total only at positions where no joker move exists in the immediate neighbourhood, and satisfy a kind of reachability condition called *P-exhaustion*. We use the model to give an analysis of a counting problem in free autonomous categories which generalizes the Triple Unit Problem.

Keywords: Full completeness; Multiplicative Linear Logic; Symmetric monoidal closed categories

1. Introduction

We aim to construct a fully complete game model for IMLL with unit, the intuitionistic multiplicative $(\otimes, \multimap, \top)$-fragment of Linear Logic (we write $\top$ for the tensor unit). The notion of full completeness [2] is best formulated in terms of a categorical model of the logic, in which formulas (or types) are denoted by objects and proofs (or terms) by maps. We say that the model $\mathbb{C}$ is *fully complete* just in case the unique functor from the relevant free category (typically the classifying category of the logic or type theory) to $\mathbb{C}$ is full and faithful. In [2] Abramsky and Jagadeesan

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PII: S0304-3975(01)00244-4

have constructed a $*$-autonomous category of games, which is a model for MLL, the classical $(-^{\perp}, \otimes, \parr)$-fragment of Linear Logic. They prove that the model is fully complete for MLL augmented by the ***Mix rule***:

$$\frac{\vdash \Gamma \qquad \vdash \Delta}{\vdash \Gamma, \Delta}.$$

Soon after the result was announced, Hyland and Ong [10] constructed a fully complete model for MLL proper, using what they call *fair games* (which invalidate the Mix rule). In both models, the treatment of units is unsatisfactory. In the former, the full completeness result really only applies to the unit-free fragment of the logic; the situation is worse in the latter, as unit cannot even be modelled (the obvious candidate is the empty game but fair games by definition have at least two moves of opposite polarity).

We stress that, for us, the categorical models of IMLL with unit are *exactly* the symmetric monoidal closed or autonomous categories. In this paper, we represent proofs of IMLL with unit by terms of a type theory called **IMLL** (which is introduced and called Autonomous Type Theory in [14]). The choice can be justified by the fact that **IMLL** is an *internal language* for autonomous categories (see [14] for details), so that the classifying category of **IMLL** is the autonomous category freely generated from (the discrete graph whose vertices are) the atomic types.

The two-person games (between P and O) we play are similar to those introduced in [1] though they are finite (no infinite plays). In Section 3, we present a new fully complete model $\mathcal{G}_e$ for IMLL without unit. Proofs are characterized by what we call *exhausting* strategies which are history-free (in the standard sense) and satisfy a kind of reachability condition called *P-exhaustion*: every P-move will eventually be played by σ by engaging some O-strategy. The exhausting strategies model is *not* the fair games model restricted to its intuitionistic part; we compare the two models in Section 6.

The rest of the paper extends the $\mathcal{G}_e$ construction, in stages, to a model $\mathcal{G}_a$ fully complete for IMLL with unit, as represented by **IMLL**. The tensor unit itself is modelled by a game called *Joker* which has a single distinguished joker move $*$. Special rules apply to $*$ which are biased towards P:

(1) P may play $*$ in response to any O-move.

(2) When O plays $*$, P is not obliged to respond, but if he does, he must do so with $*$.

Correspondingly the exhausting strategies (for P), thus extended, are neither deterministic nor total, though these properties must still hold *locally* whenever the joker move is not the last move and no joker move is available to P at that point. We introduce such strategies, called *conditionally exhausting*, in Section 8. Conditionally exhausting strategies compose and form a category $\mathcal{G}_{ce}$ (though it is not autonomous) which we show is fully complete for an intermediate type theory called $\mathbf{IMLL}^{\flat}$ in Sections 7 and 8. In Section 9 we show that by quotienting the homsets of the category $\mathcal{G}_{ce}$ by an appropriate equivalence relation, we obtain an autonomous category

$\mathcal{G}_a$ which is fully complete for **IMLL**. To our knowledge, this is the first such game model.

We conclude the paper by considering an application in Section 10. We use the model to give an analysis of what we call the *Tower of Units Problem*, which is a counting problem in free autonomous categories that generalizes the Triple Unit Problem.

1.1. On related work

Full completeness results for the multiplicative fragment of Linear Logic abound. They fall into two groups. The first consists of fully complete models of $\mathrm{MLL} + \mathrm{MIX}$ e.g. [2, 16, 6], all of which are isoMIX categories (rather than MIX categories) in the sense of [7] i.e. the tensor unit $\top$ is isomorphic to the "par unit" $\bot$. This means that there is a map from $(a \multimap \top) \multimap \top$ to a, but the corresponding sequent is not provable in IMLL (with unit), so these models are not fully complete. The second group consists of models of MLL proper (without MIX) e.g. [10, 17, 8, 3], but in each case, full completeness has been proved for the unit-free fragment only. In the last three both units are introduced, but no claim has been made with respect to the fragment with units. In our view, the respective full completeness results for MLL with units, if valid, are likely to require complicated proofs see e.g. [5, 14].

2. Autonomous categories of games

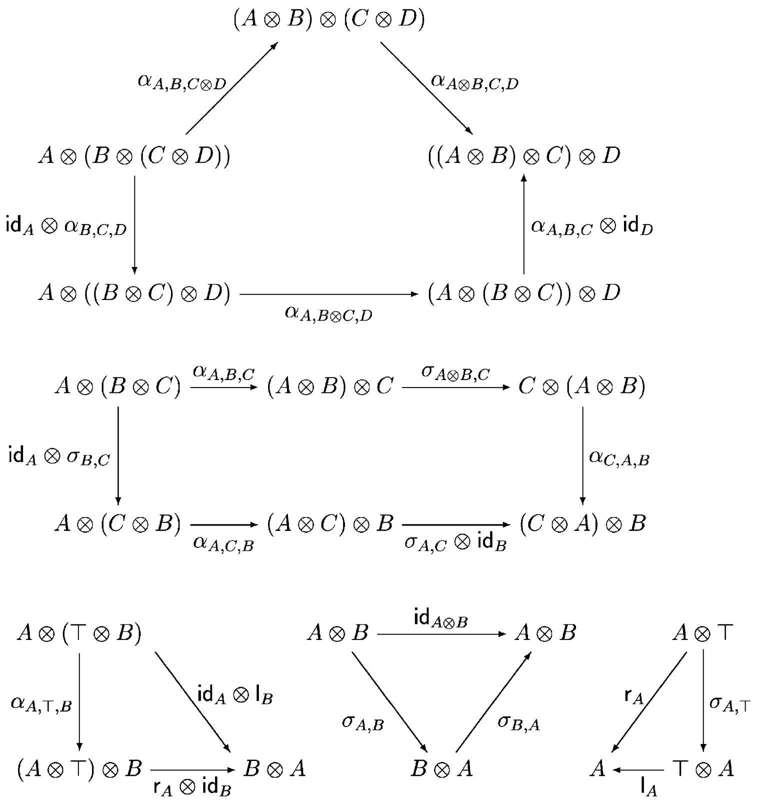
In this section, we set out the basics of game semantics for Intuitionistic Multiplicative Linear Logic (IMLL) [9]. We introduce four categories of games and strategies, which we will refine in subsequent sections to give a fully complete model for IMLL with unit.

Recall that a *symmetric monoidal category* $\mathscr{C}$ is a category equipped with a bifunctor $\otimes : \mathscr{C} \times \mathscr{C} \to \mathscr{C}$, an object $\top$ called tensor unit [3] and four isomorphisms

$$\begin{array}{rrll} \alpha_{A,B,C}: & A \otimes (B \otimes C) & \to (A \otimes B) \otimes C & \textit{associativity} \\ \sigma_{A,B}: & A \otimes B & \to B \otimes A & \textit{symmetry} \\ \mathsf{l}_A: & \top \otimes A & \to A & \textit{left unit} \\ \mathsf{r}_A: & A \otimes \top & \to A & \textit{right unit} \end{array}$$

defined for and natural in all objects A, B and C such that the following diagrams commute:

[3] We follow Barr [4] in writing the tensor unit as $\top$ instead of the more traditional I.



Moreover $\mathsf{l}_\top$ and $\mathsf{r}_\top : \top \otimes \top \to \top$ are required to coincide. We say that $\mathbb{C}$ is *symmetric monoidal closed* or *autonomous* just in case for each $A \in \mathbb{C}$, the functor $(-) \otimes A$ has a specific right adjoint $A \multimap (-)$.

We shall consider two-player games between P (Proponent) and O (Opponent). Every play is started by O, and thereafter it alternates between P and O. We construct new games from old using the standard tensor and linear implication constructions.

Definition 1. A ***game*** G is a triple $\langle M_G, \lambda_G, P_G \rangle$ where

(i) M_G is a set of moves

(ii) $\lambda_G : M_G \to \{O, P\}$ partitions moves into those that O can make or *O-moves*, and those that P can make or *P-moves* (we will write M_G^O, M_G^P for the set of O-moves and P-moves of G, respectively, or simply M^O and M^P whenever G is understood)

(iii) P_G is a prefix-closed set of finite alternating sequences of moves from M_G, each beginning with an O-move; we call elements of P_G *positions* or *plays* of G.

We say that a game G is *finite* if both M_G and P_G are finite sets. Henceforth we shall assume that our games are finite. For example $\langle \emptyset, \emptyset, \{\varepsilon\} \rangle$ (where ε is the empty sequence) is a game, which we shall call the *empty game*, written $\emptyset$. We interpret (non-unit) atomic types a as single-move games G_a, which we shall also call ***atomic***, defined as follows

$$G_a = \langle \{a\}, \{(a, O)\}, \{\varepsilon, a\} \rangle.$$

For sets S and T, we fix a representation of disjoint union $S + T$ as

$$S + T = \{(0, s):\ s \in S\} \cup \{(1, t):\ t \in T\}$$

and for functions $f_i : P_i \to Q$ where $i = 0, 1$, we write $[f_0, f_1] : P_0 + P_1 \to Q$ for the canonical function. For a game G, we write $M_G^{\circledast}$ to mean the set of finite alternating sequences of moves from M_G. Given games A and B, we define the ***tensor game*** $A \otimes B$ as follows:

$$M_{A\otimes B} = M_A + M_B,$$

$$\lambda_{A\otimes B} = [\lambda_A, \lambda_B],$$

$$P_{A\otimes B} = \{s \in M_{A\otimes B}^{\circledast}:\ s \upharpoonright A \in P_A, s \upharpoonright B \in P_B\},$$

where we write $s \upharpoonright A$ to mean the subsequence of s consisting only of moves from A. Note that it is a consequence of the definition that every $s \in P_{A\otimes B}$ satisfies

O-Switching Condition: for each pair of consecutive moves mm' in s, if m and m' are from different components (i.e. one is from A the other from B), then m' is an O-move.

The ***linear implication game*** $A \multimap B$ is defined as follows:

$$M_{A\multimap B} = M_A + M_B,$$

$$\lambda_{A\multimap B} = [\overline{\lambda_A}, \lambda_B] \quad (\text{where } \bar{P} = O \text{ and } \bar{O} = P),$$

$$P_{A\multimap B} = \{s \in M_{A\multimap B}^{\circledast}:\ s \upharpoonright A \in P_A, s \upharpoonright B \in P_B\}.$$

It follows from the definition that every $s \in P_{A\multimap B}$ satisfies the *P-Switching Condition* (i.e. only P can switch component).

A *P-strategy*, or simply ***strategy***, for a game G is a non-empty, prefix-closed subset σ of P_G such that for any even-length s, if $s \in \sigma$ and $sm \in P_G$ then $sm \in \sigma$. If for every odd-length $s \in \sigma$, there is some m such that $sm \in \sigma$, we say that σ is ***total***. We say that σ is ***deterministic*** if for any odd-length s, if $sa \in \sigma$ and $sb \in \sigma$ then $a = b$. Note that strategies are *not* assumed to be deterministic or total, unless explicitly stated.

For any games A_1, A_2 and A_3 we define $\mathscr{L}(A_1, A_2, A_3)$ to be the set of finite sequences s of moves from $M_{A_1} + M_{A_2} + M_{A_3}$ such that for any pair of consecutive moves mm' in s, if $m \in M_{A_i}$ and $m' \in M_{A_j}$ then $|i - j| \leqslant 1$. Take strategies σ and τ for games $A \multimap B$ and $B \multimap C$, respectively. We define the composite $\sigma ; \tau$ of σ and τ as:

$$\sigma ; \tau = \{s \upharpoonright (A, C):\ s \in \mathscr{L}(A, B, C) \wedge s \upharpoonright (A, B) \in \sigma \wedge s \upharpoonright (B, C) \in \tau\}.$$

This is the by now standard (CSP parallel composition plus hiding) notion of composition of strategies [2]. We distinguish four classes of strategies:
(1) strategies,
(2) deterministic strategies,
(3) total strategies,
(4) deterministic and total strategies.
All four classes compose in the sense that if σ and τ are strategies of the same class $\mathscr{S}$ for $A \multimap B$ and $B \multimap C$, respectively, then the composite $\sigma ; \tau$ for $A \multimap C$ is a strategy of the class $\mathscr{S}$. For composition of total strategies, the assumption of finiteness is essential.

We can define an autonomous category whose objects are finite games and whose maps $A \to B$ are given by strategies for the game $A \multimap B$. The categorical tensor and linear implication constructions (on objects) are just the corresponding constructions on games. Observe that the games $C \otimes A \multimap B$ and $C \multimap (A \multimap B)$ are "identical" modulo a renaming of moves across the bijection $(M_C + M_A) + M_B \cong M_C + (M_A + M_B)$. The tensor constructor can be lifted to a bifunctor. For strategies $\sigma_1 : A_1 \to B_1$ and $\sigma_2 : A_2 \to B_2$, $\sigma_1 \otimes \sigma_2 : A_1 \otimes A_2 \to B_1 \otimes B_2$ is the strategy that plays according to σ_1 or σ_2 in a "non-communicating way":

$$\sigma_1 \otimes \sigma_2 = \{ s \in P_{A_1 \otimes A_2 \multimap B_1 \otimes B_2} \mid s \upharpoonright (A_1, B_1) \in \sigma_1, s \upharpoonright (A_2, B_2) \in \sigma_2 \}.$$

It is easy to see that $(\sigma_1 ; \tau_1) \otimes (\sigma_2 ; \tau_2) = (\sigma_1 \otimes \sigma_2);(\tau_1 \otimes \tau_2)$ and $\mathsf{id}_{A \otimes B} = \mathsf{id}_A \otimes \mathsf{id}_B$. For this category, the tensor unit is the empty game $\emptyset$; and the canonical isos $\mathsf{l}_G, \mathsf{r}_G$, $\alpha_{A,B,C}$ and $\sigma_{A,B}$ are just the obvious "copycat" strategies.

Theorem 2. *The category has three symmetric monoidal closed subcategories whose maps are, respectively, given by deterministic, total, and deterministic and total strategies. The symmetric monoidal closed structure is inherited from the larger category.*

Definition 3. Fix a set universe $\mathscr{U} = \{a, b, c, \ldots\}$ of *tokens* (one for each atomic type). We write $\mathscr{G}$ for the autonomous category whose objects are games that are freely constructed from the atomic games $G_a, G_b, G_c \ldots$ (one for each token from $\mathscr{U}$), using tensor, linear implication and the nullary constructor $\emptyset$, and whose maps are given by strategies as defined in the preceding. We shall refer to objects of $\mathscr{G}$ as ***free games***. We write $\mathscr{G}_d, \mathscr{G}_t$ and $\mathscr{G}_{d,t}$ for the three subcategories whose objects are free games, and whose maps are respectively given by deterministic, total, and deterministic and total strategies. Note that every move m of a free game arises from some unique token from $\mathscr{U}$, which we shall refer to as m's ***token***.

Notation 4. In the following we shall abuse notation and write the atomic game G_a simply as a, so that the letters a, b, c, etc. can mean either atomic types (to be introduced in Section 3), or atomic games, or their respective singleton moves, depending on the context. E.g. we shall write $G_a \otimes (G_b \multimap G_b) \multimap G_a$ simply as $a \otimes (b \multimap b) \multimap a$.

3. Exhausting strategies and IMLL$^-$

The category $\mathscr{G}_{d,t}$ is too rich to be fully complete for the intuitionistic $(\multimap, \otimes)$-fragment of Linear Logic [9]. Consider the game

$$a \otimes (b \multimap b) \multimap a.$$

The "copycat" strategy for the game is total but the corresponding sequent $a \otimes (b \multimap b) \vdash a$ is not provable in Linear Logic. We attribute this to the fact that there is some P-move of the game which is never played by the strategy. This leads to a new notion of strategies.

Definition 5. First we say that a strategy σ is ***O-oriented*** if there is a partial function $f : M^P \rightharpoonup M^O$ such that for every P-move m, if the odd-length $sm' \in \sigma$ then

$$f(m) = m' \Leftrightarrow sm'm \in \sigma;$$

such an f is called a *predecessor* function of σ. We write $\sigma = \sigma_f$, and say that f ***generates*** σ, if f is the *least* such function.

We can think of O-oriented strategies as a kind of non-deterministic generalization of history-free strategies in that for each P-move m, whenever m is played, the O-move which triggers it (i.e. m's predecessor) is unique. If $f(m)$ and m have the same token for all m, we say that f is ***token-reflecting***.

Definition 6. A strategy σ for a game is said to be ***exhausting*** if σ is O-oriented and generated by some token-reflecting bijection $f : M^P \to M^O$ (i.e. $\sigma = \sigma_f$).

Although f is a function from M^P to M^O, its bijectivity here implies that σ_f is deterministic and indeed history-free in the standard sense. Further it follows from the definition that exhausting strategies are total and satisfy the ***P-exhaustion condition***

$$\forall m \in M^P. \exists k \geqslant 0. \exists m_1, \ldots, m_k \in M^P. f(m_1)\, m_1 \cdots f(m_k)\, m_k\ f(m)\, m \in \sigma$$

(for if for some m, there is no position ending with $f(m)$ in σ, then f, which is the least predecessor function by assumption, cannot be total).

We begin with a definition of the *enabling* (or justification) relation [11, 19] between moves of a free game or, by analogy, between atoms of an **IMLL**$^-$ formula. The set $in(G)$ of initial moves of a free game G is defined by recursion as follows:

$$in(G_a) = \{a\},$$

$$in(G_1 \otimes G_2) = in(G_1) \cup in(G_2),$$

$$in(G_1 \multimap G_2) = in(G_2).$$

Given moves x and y of a free game G, we say that y ***enables*** x if G has the form $C[X \multimap Y]$ for some one-holed context C, some free games X and Y such that $x \in in(X)$ and $y \in in(Y)$.

Definition 7. A position of a free game G is ***shortsighted*** if every O-move in it is enabled by the P-move that precedes it.

We claim that for any position sp (of a free game) ending in a P-move p, if p enables an O-move o, then spo is a position. This is because from the definition of the linear implication game, we know that o can be played at sp provided it has not been used previously in s, which is indeed the case, as the next lemma shows:

Lemma 8. *Let G be a free game. Let p_1, p_2 be P-moves such that an O-move o is enabled by both p_1 and p_2. G has no position of the following shape*: $\cdots p_1 o \cdots p_2 \cdots$ *if O plays shortsightedly.*

Proof. W.l.o.g. assume $G = C_1^-[O \multimap C_2^-[P_1 \otimes P_2]]$, where o, p_1, p_2 are initial moves of O, P_1, P_2, respectively. Note that between p_1 and p_2 an O-move from P_1 must be made. Take the first such. In a shortsighted position it has to be preceded by an enabling P-move. This enabling P-move must be in P_1 as well, which is a contradiction. □

Thus for exhausting strategies, P-exhaustion is witnessed by shortsighted positions, i.e. for any P-move m there is a shortsighted position that reaches m.

Lemma 9. *Suppose σ is an O-oriented strategy for a free game so that $\sigma = \sigma_f$ for some f. Then f is a bijection if and only if σ is deterministic, total and satisfies P-exhaustion.*

Proof. Observe that for any game f is injective if and only if σ_f is deterministic, and f is a (total) function if and only if σ_f satisfies P-exhaustion. Further, if f is surjective, σ_f is total. Finally, if σ_f is total and P-exhausting and the game is free, then f is surjective. To prove the last assertion, consider an O-move m^O. If it is initial, then we must have $f(m) = m^O$ for some P-move m, because σ_f is total. If m^O is not initial, there is a P-move m enabling it. Because σ_f satisfies P-exhaustion, by the preceding remark there exists a shortsighted position $sm \in \sigma_f$ and $smm^O \in \sigma_f$. By totality of σ_f and the definition of O-oriented strategies, there must exist a P-move m with $f(m) = m^O$. □

Note that there may be several ways for an exhausting strategy to reach a particular P-move. For example, the token-reflecting strategy for

$$(a \multimap b \otimes c) \otimes a \otimes (b \multimap d) \multimap c \otimes d$$

has two maximal positions that end in the P-move (whose token is) a.

As we shall see, the subcategory $\mathcal{G}_e$ (of $\mathcal{G}_{d,t}$) whose objects are *non-empty* free games and whose maps are given by exhausting strategies is a fully complete model of **IMLL**$^-$. The composition of exhausting strategies is defined by "inheriting" the standard notion of composition from $\mathcal{G}_{d,t}$ as given in Section 1. By first establishing definability (i.e. every exhausting strategy is the denotation of a cut-free **IMLL**$^-$ proof-term), we can prove that the composition of exhausting strategies as given is well defined by a syntactic argument. At the moment, we do not have a *direct* "syntax-free" proof.

Remark 10. The difficulty with proving compositionality directly is already apparent in a special case of composition. Take exhausting strategies σ and τ for B and $B \multimap C$, respectively. To prove that the "composite" $\sigma\,;\tau$ enjoys P-exhaustion, for any P-move m in C, we have to find a "witnessing position" in P_C that reaches it. Now τ gives us one such p (say) but it is in $P_{B\multimap C}$. What we then need to show is that p is played out in the interaction $\sigma;\tau$. But the O-moves of B in the interaction are dictated by σ during the composition. Thus what one needs to show is that P-exhaustion holds even when O behaves in a particular way on B, and this seems difficult.

3.1. The type theory IMLL$^-$

We give a brief introduction to the type theory **IMLL**$^-$, which is a "unit-less" fragment of the Autonomous Type Theory [14] (the minus sign "−" indicates the absence of the tensor unit). There are three kinds of judgements:
- *typing*: $\Gamma \vdash s : A$;
- *equality*: $\Gamma \vdash s = t : A$;
- *congruence*: $\Gamma \vdash s \sim t : A$;

where Γ, Δ, etc. range over typing contexts, which are finite sequences of variable-type pairs (of the form $x : A$) in which no variable may occur more than once. The types (ranged over by A, B, C, etc.) are built up from a set $\{a, b, c, \ldots\}$ of atomic types by the tensor $\otimes$ and linear function space $\multimap$ constructors. The valid typing judgements are defined by the rules in Fig. 1. If $\Gamma \vdash s : A$ is valid, we say that s is well-typed *proof-term* or simply *term*. By a *sequent* we mean expressions of the form $\Gamma \vdash A$ where $\Gamma = C_1, \ldots, C_n$ here (by abuse of notation) is a finite sequence of types. If $\Gamma \vdash s : A$ is provable, we say that s *inhabits* the sequent $\Gamma \vdash A$.

A central feature of our approach is the representation of the categorical composition, or equivalently the (**cut**) rule in the type theory, by *explicit substitution*; indeed the standard substitution has no place in our approach at all. In our type theory it is a property of well-typed terms that any variable which occurs in a term (whether bound or free) does so exactly once. In $s\{t/x^A\}$, which we call an ***explicit substitution*** (term), the free occurrence of x in s is bound. We introduce two *let-constructs* as terms witnessing the left-introduction rules of the respective type constructors $\otimes$ and $\multimap$. In the ***tensor-let*** $\langle z^{A\otimes B}/x^A \otimes y^B\rangle s$, the free occurrences of x and y in s (it can be shown that there is exactly one each if s is well-typed) are bound by the let-

(id-atom) $x : a \vdash x^a : a \quad (a \text{ atomic})$

(exch) $$\frac{\Gamma, x : A, y : B, \Delta \vdash s : C}{\Gamma, y : B, x : A, \Delta \vdash s : C}$$

(cut) $$\frac{\Gamma, x : A \vdash s : B \qquad \Delta \vdash t : A}{\Gamma, \Delta \vdash s\{t/x^A\} : B}$$

(⊗-l) $$\frac{x : A, y : B, \Gamma \vdash s : C}{z : A \otimes B, \Gamma \vdash \langle z^{A \otimes B}/x^A \otimes y^B \rangle s : C}$$

(⊗-r) $$\frac{\Gamma \vdash s : A \qquad \Delta \vdash t : B}{\Gamma, \Delta \vdash s \otimes t : A \otimes B}$$

(⊸-l) $$\frac{\Gamma \vdash s : A \qquad y : B, \Delta \vdash t : C}{z : A \multimap B, \Gamma, \Delta \vdash \langle z^{A \multimap B}, s/y^B \rangle t : C}$$

(⊸-r) $$\frac{\Gamma, x : A \vdash s : B}{\Gamma \vdash \boldsymbol{\lambda} x^A.s : A \multimap B}$$

Fig. 1. Rules defining the valid typing judgements of **IMLL**$^-$.

construct. In the ***lambda-let*** construct $\langle z^{A \multimap B}, s/y^B \rangle t$, the free occurrence of y in t is bound.

The valid equality judgements are defined by three axioms (in addition to the standard equality rules in Fig. 4):

(id) $\Gamma \vdash x^a s/x^a = s \ : \ a \quad (a \text{ is an atomic type})$

$(\otimes)$ $\Gamma \vdash (\langle z^{A\otimes B}/x \otimes y \rangle s)\{u \otimes v/z\} = (s\{u/x^A\})\{v/y^B\} : C,$

$(\multimap)$ $\Gamma \vdash (\langle z^{A \multimap B}, s/y^B \rangle t)\{\boldsymbol{\lambda} x^A.u/z\} = t\{u\{s/x^A/y^B\}\} : C.$

Terms that are congruent are defined to be equal in the theory:

(cong) $$\frac{\Gamma \vdash s \sim t : A}{\Gamma \vdash s = t \ : \ A}.$$

There are two commutation congruence axioms (in addition to the standard congruence rules in Fig. 5):

(π-cong) $\Gamma \vdash \pi C[t] \sim C[\pi t] \ : \ A,$

(σ-cong) $\Gamma \vdash C[t]\sigma \sim C[t\sigma] \ : \ A,$

where we let π range over the let-constructs, namely $\langle z^{A\otimes B}/x \otimes y \rangle-$ and $\langle z^{A \multimap B} s/y \rangle-$, and σ range over the explicit substitution constructs $-\{t/x^A\}$ (so that π is viewed as a prefix and σ a postfix operator). We let C, D, E etc. range over one-holed contexts, or simply *contexts*, defined by recursion as follows:

$$C ::= [\text{-}] \quad | \quad s\{C/x^A\} \quad | \quad C\{s/x^A\} \quad | \quad s \otimes C \quad | \quad C \otimes s \quad | \quad \boldsymbol{\lambda} x^A.C$$
$$| \quad \langle z^{A\otimes B}/x \otimes y\rangle C \quad | \quad \langle z^{A\multimap B}, C/y^B\rangle s \quad | \quad \langle z^{A\multimap B}, s/y^B\rangle C.$$

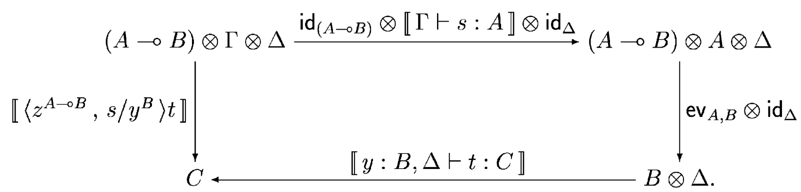
Standardly we write $C[s]$ to mean "the capture-permitting substitution of s for the hole in C". Each congruence axiom above is required to satisfy the side condition, called ***strong typability***, that the expressions on both sides of $\sim$ must be well-typed terms of the same declared type. The effect of the axioms is simply that the let-construct π- or the explicit substitution construct -σ, viewed as a variable binder, may "float across the term" and is free to occupy any position in the term *provided* that typability is maintained. For example, the binder $-\{s/x^A\}$ in $(\boldsymbol{\lambda} y^B.x \otimes y)\{s/x^A\}$ is permitted to park itself adjacent to x, as in $\boldsymbol{\lambda} y^B.(x\{s/x^A\} \otimes y)$ (as the two terms are both well-typed, they are congruent to each other by ($\boldsymbol{\sigma}$-cong)), but not adjacent to y, as in $\boldsymbol{\lambda} y^B.(x \otimes (y\{s/x^A\})$ which is not well-typed.

3.2. *Interpretation of* **IMLL**$^-$ *in* $\mathscr{G}_{d,t}$ *and the subcategory* $\mathscr{G}_e$

It is shown in [14] that there is a canonical interpretation of the typing judgements $\Gamma \vdash s : A$ of **IMLL**$^-$ in *any* autonomous category $\mathbb{C}$ such that
(i) if $\Gamma \vdash s : A$ is provable then $[\![\Gamma \vdash s : A]\!]$ is a $\mathbb{C}$-map $[\![\Gamma]\!] \to [\![A]\!]$, with $[\![\Gamma]\!] = (([\![C_1]\!] \otimes [\![C_2]\!]) \otimes [\![C_3]\!]) \otimes \cdots \otimes [\![C_n]\!]$ where $\Gamma = x_1 : C_1, \ldots, x_n : C_n$, and
(ii) if $\Gamma \vdash s = t : A$ is provable then $[\![\Gamma \vdash s : A]\!]$ and $[\![\Gamma \vdash t : A]\!]$ are equal as $\mathbb{C}$-maps.
Henceforth we shall write $[\![\Gamma \vdash s : A]\!]$ to mean the denotation of the sequent in the autonomous category $\mathscr{G}_{d,t}$. In the following, by a *cut-free term* we mean a term that does not have any explicit substitution subterms.

Proposition 11. *If* $\Gamma \vdash s : A$ *is* **IMLL**$^-$*-provable and* s *is cut-free then its denotation* $[\![\Gamma \vdash s : A]\!]$ *in* $\mathscr{G}_{d,t}$ *is given by an exhausting strategy.*

Proof. We refer to [14] for the definition of $[\![\Gamma \vdash s : A]\!]$ in an autonomous category. The Proposition is more or less obvious. It is trivial to see that currying and tensor preserve P-exhaustion. But strictly speaking, composition is still needed to model ($\multimap$-**l**). In the following we show how the sequent $z : A \multimap B, \Gamma, \Delta \vdash \langle z^{A\multimap B}, s/y^B\rangle t : C$ is interpreted (in the diagram we do not distinguish notationally a finite sequence Γ of types from its denotation $[\![\Gamma]\!]$, but no confusion should arise):



Fortunately the composition required is quite innocuous as the strategies $[\![y:B,\Delta \vdash t:C]\!]$ and $[\![\Gamma \vdash s:A]\!]$ do not really interact. The composite is in fact given simply by the (disjoint) union of the generating functions of the respective component strategies: it is a harmless interleaving of a kind similar to $[\![y:B,\Delta \vdash t:C]\!] \otimes [\![\Gamma \vdash s:A]\!]$. □

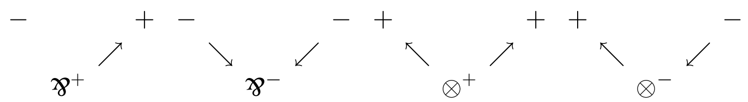
4. Essential nets and correctness criterion

This section is about essential nets [15], which are an intuitionistic variant of Girard's proof nets [9]. We say that an **IMLL**$^-$ (or **IMLL**$^\flat$ or **IMLL**) sequent (or typing judgement or formula) is ***linearly balanced*** if every non-unit atomic type that occurs in it does so exactly twice and the occurrences are of opposite polarities. Linearly balanced sequents are not necessarily provable (e.g. $\vdash (a \multimap b) \otimes (b \multimap a)$), nor are provable sequents necessarily linearly balanced (e.g. $a, a \vdash a \otimes a$). However by a simple induction over the proof rules, we can readily see that if $\Gamma \vdash s:A$ is provable then s determines a bijection between positive and negative occurrences of (non-unit) atomic types in the sequent, which we shall henceforth refer to as a ***linkage*** for the sequent.

Any linearly balanced **IMLL**$^-$ formula A can be transformed to a variant MLL formula $\ulcorner A \urcorner^+$ which is constructed from polarized atoms (a^+, a^-, b^+, b^-, etc.) and polarized connectives ($\parr^+, \parr^-, \otimes^+$ and $\otimes^-$) by the following rules:

$$\begin{array}{ll} \ulcorner a \urcorner^+ = a^+, & \ulcorner a \urcorner^- = a^-, \\ \ulcorner A \multimap B \urcorner^+ = \ulcorner A \urcorner^- \parr^+ \ulcorner B \urcorner^+, & \ulcorner A \multimap B \urcorner^- = \ulcorner A \urcorner^+ \otimes^- \ulcorner B \urcorner^- \\ \ulcorner A \otimes B \urcorner^+ = \ulcorner A \urcorner^+ \otimes^+ \ulcorner B \urcorner^+, & \ulcorner A \otimes B \urcorner^- = \ulcorner A \urcorner^- \parr^- \ulcorner B \urcorner^-. \end{array}$$

We construct a directed graph $\mathscr{E}(A)$ from the syntactic tree of $\ulcorner A \urcorner^+$, augmented by axiom links, by orienting axiom links from a^+ to a^- (note that A is assumed to be linearly balanced) and drawing in directed edges for the polarized connectives according to the directional rules as follows:



Note that there is no edge between a $\parr^+$-node and its negative left "child", which we shall call its *sink*. We call $\mathscr{E}(A)$ the ***essential net*** of A. See Fig. 2 for an example of an essential net.

Theorem 12 (Lamarche). *A linearly balanced* **IMLL**$^-$ *formula A is provable if and only if $\mathscr{E}(A)$ satisfies the following correctness criteria*:

(i) *acyclicity*,

(ii) ***Condition L***: *for every $\parr^+$-node p, every path from the root that reaches p's sink passes through p.*

If $\mathscr{E}(A)$ satisfies the two conditions in Theorem 12, we call it a *correct* essential net.

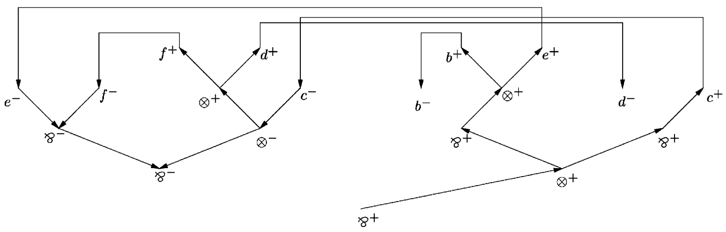


Fig. 2. The essential net of $(e \otimes f) \otimes (f \otimes d \multimap c) \multimap (b \multimap b \otimes e) \otimes (d \multimap c)$.

Remark 13. (i) In this paper, essential nets are by definition directed graphs of a certain kind constructed from *linearly balanced* IMLL formulas. For such nets (but not in general—consider $\mathscr{E}(\vdash a \multimap b)$), it is straightforward to prove that the *reachability condition*, i.e. every node is reachable from the root, is a consequence of acyclicity.

(ii) A version of the theorem (which includes the reachability condition) first appeared in [15] which has not been published. Our proof (for the completeness part) is based on a reduction of formulas to what we call *regular formulas*, and an analysis of the essential nets of such formulas. Regular reduction is an intuitionistic variant of the reduction to simple sequents introduced in [2] and modified to semi-simple sequents in [10]. Note however that the normal forms in our case (i.e. the regular formulas) are more complicated than simple or semi-simple sequents.

Proof of Theorem 12. We devote the rest of this section to a proof of the theorem. The soundness ($\Rightarrow$) part can be proved by a straightforward induction over the rules of the IMLL Sequent Calculus (less the cut rule). For the completeness part, it suffices to prove: for any linearly balanced IMLL sequent of the form $\vdash \Theta$, if $\mathscr{E}(\Theta)$ is correct then $\vdash \Theta$ is provable. We perform induction on the number of atoms for *regular* formulas which are defined by recursion over the following rules:

$$T ::= \underbrace{T^- \otimes \cdots \otimes T^-}_{m} \multimap \underbrace{\theta \otimes \cdots \otimes \theta}_{n} \;\mid\; \underbrace{\theta \otimes \cdots \otimes \theta}_{n},$$

$$T^- ::= \underbrace{T \otimes \cdots \otimes T}_{m} \multimap \theta \mid \theta,$$

where θ ranges over atoms and $m, n \geqslant 1$.

We can transform an irregular formula Θ to a finite set of regular ones by replacing Θ by Θ_1, Θ_2 repeatedly, using the following reduction rules:

Θ	Θ_1	Θ_2
$C^+[A \otimes (B \multimap C)]$	$C^+[(A \multimap B) \multimap C]$	$C^+[B \multimap A \otimes C]$
$C^+[(B \multimap C) \otimes A]$	$C^+[(A \multimap B) \multimap C]$	$C^+[B \multimap A \otimes C]$
$C^-[A \multimap B \otimes C]$	$C^-[(A \multimap B) \otimes C]$	$C^-[(A \multimap C) \otimes B]$
$C[A \multimap (B \multimap C)]$	$C[A \otimes B \multimap C]$	

where C is a one-holed *type* context, C^+ is a context whose hole occurs in a positive position, and C^- is a context whose hole occurs in a negative position. First we note the following (see [2] for a proof):

Proposition 14. *For types Θ, Θ_1 and Θ_2 as given in the first three reduction rules, if $\vdash \Theta_1$ and $\vdash \Theta_2$ then $\vdash \Theta$.* □

The reduction process is strongly normalising. A measure on types can be defined as follows:

$$\mu(\theta) = 2,$$

$$\mu(D \otimes^+ E) = \mu(D \multimap^- E) = \mu(D) \times \mu(E),$$

$$\mu(D \otimes^- E) = \mu(D \multimap^+ E) = \mu(D) + \mu(E).$$

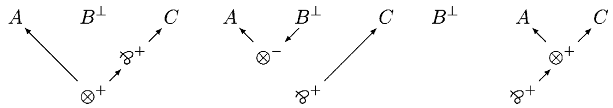
The measure is strictly decreasing for the first three reduction rules; it is invariant for the last but note that the number of occurrences of $\multimap$ strictly decreases in that case. Plainly the normal forms of the reduction are regular by design.

Thus the reduction terminates with a finite set of regular formulas. In fact, each of these formulas is guaranteed to induce a correct essential net, which we know because of the following lemma.

Lemma 15. *For Θ, Θ_1 and Θ_2 as given in the first reduction rules, if $\mathscr{E}(\Theta)$ is correct, then so is $\mathscr{E}(\Theta_1)$ and $\mathscr{E}(\Theta_2)$.*

Proof. We show that the correctness criteria are preserved by a case-by-case analysis of the reduction rules.

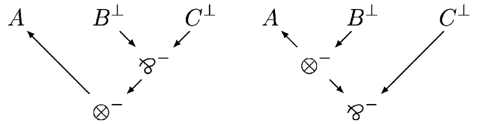
(i), (ii): The two cases are symmetric so we consider the first one only. In the picture below we show the nodes of $\mathscr{E}(\Theta)$ that are affected by the reduction and their new arrangement in $\mathscr{E}(\Theta_1)$ and $\mathscr{E}(\Theta_2)$, respectively.



If there were a cycle in $\mathscr{E}(\Theta_1)$, it would have to involve the $\otimes^-$ node as otherwise it would be in $\mathscr{E}(\Theta)$. However, this would mean that Condition L for $\mathscr{E}(\Theta)$ is violated. Acyclicity of $\mathscr{E}(\Theta_2)$ follows from acyclicity of $\mathscr{E}(\Theta)$.

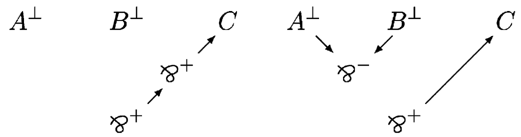
Note that Condition L must be satisfied for the $⅋^+$ node in $\mathscr{E}(\Theta_1)$ because $\mathscr{E}(\Theta)$ satisfies Condition L. If it is broken for some other $⅋^+$ node, then the path that violates it would have to pass through the $\otimes^-$ node and, by previous remark, through the $⅋^+$ node in the picture. Then there would be a path in $\mathscr{E}(\Theta)$ passing through the $\otimes^+$ node that also violates Condition L. It is straightforward to see that for $\mathscr{E}(\Theta_2)$ Condition L holds too.

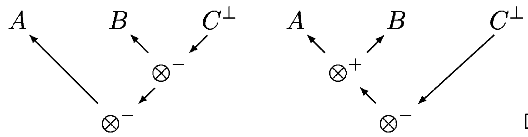
(iii): Because of symmetry, it suffices to consider just one of Θ_1 and Θ_2. We draw the distinctive nodes of $\mathscr{E}(\Theta)$ and $\mathscr{E}(\Theta_1)$, respectively.



Note that the reachability relation between A, $B^{\perp}$ and $C^{\perp}$ of the graph on the right is included in that of the left graph which accounts for the preservation of acyclicity and Condition L.

(iv): There are two cases here depending on the polarity of the context. The pictures below show the edges of $\mathscr{E}(\Theta)$ (on the left) which will be modified by the reduction and their new arrangement in $\mathscr{E}(\Theta_1)$ (on the right). The pictures speak for themselves.





□

Thus we can prove the completeness part of the theorem by induction for the special case where Θ is regular. We can then extend the result to the general case by appealing to Proposition 14. Before presenting the inductive argument, we prove a useful technical lemma.

Lemma 16. (i) (Splitting). *Suppose for each i, $S_i = T_i \multimap a_i$ or $S_i = a_i$ where a_i is an atom and $\mathscr{E}(S_1 \otimes \cdots \otimes S_n \multimap M_1 \otimes M_2)$ is correct. Then there exists a (unique) partition of $\{1, \ldots, n\}$ into X_1 and X_2 such that $\mathscr{E}(\bigotimes_{x \in X_i} T_x \multimap M_i)$ is correct.*

(ii) *Suppose the essential net of a regular formula of the form $T_1^- \otimes \cdots \otimes T_n^- \multimap a$ is correct. Then for some T' and some i, we have $T_i^- = T' \multimap a$ and the essential net for*

$$T_1^- \otimes \cdots \otimes T_{i-1}^- \otimes T_{i+1}^- \otimes \cdots \otimes T_n^- \multimap T'$$

is again correct.

Proof. (i) Let us construct an undirected graph with vertices $T_1, \ldots, T_n, M_1, M_2$. There is an edge between T_i and some other vertex V if V contains the positive occurrence of a_i. Thus each T_i is the end of precisely one link. Now take X_i to be the set of indices of T_j's in the same component as M_i. By previous remarks X_1 and X_2 are disjoint. By Remark 13 each $a_i^{\perp}$ can be reached from the root in $\mathscr{E}(S_1 \otimes \cdots \otimes S_n \multimap M_1 \otimes M_2)$. Since the only way of entering (the part corresponding to) S_i in the essential net is through the axiom link between $a_i^{\perp}$ and a_i, if a_i can be reached via M_j, T_i is in the same component as M_j. Moreover, each path in the essential net that visits M_j can only pass through the nodes corresponding to the T_i's lying in the component containing M_j. Therefore the partition yields two correct essential nets.

(ii) Recall that each T_i^- is of shape $T^i \multimap x$ and the only way of entering the part corresponding to T_i^- in the essential net is via the axiom link of $x^{\perp}$. The first node reachable from the root is a so the next one will be $a^{\perp}$. As it must be part of some T_i^-, that T_i^- is of shape $T' \multimap a$. The transformed essential net must be correct as its structure is inherited from the old one. □

To conclude the proof, it remains to give the inductive argument for the case where Θ is regular. If Θ has only two atoms, we have $\Theta = a \multimap a$, so the corresponding sequent is IMLL-provable. The preceding Lemma enables us to reduce the size of regular formulas. Note that T' has the shape $T \otimes \cdots \otimes T$ and we can use Lemma 16(i) to consider formulas of the shape $T^- \otimes \cdots \otimes T^- \multimap T$. Then T has the shape $T^- \otimes \cdots \otimes T^- \multimap \theta \otimes \cdots \otimes \theta$ and by using the last regular transform, we get a truly regular formula.

5. Definability and cull completeness

In this section we prove that every exhausting strategy for a free game is the denotation of a term inhabiting the associated typing judgement, and any two such terms are provably equal. Our definability proof uses essential nets. We show that there is a correspondence between shortsighted positions of a free game G and paths (projected onto leaf nodes) from the root of the corresponding essential net provided G, *qua* formula, is provable. As a pleasing corollary of the definability result, we show that exhausting strategies compose and that $\mathscr{G}_e$ is a well-defined subcategory of $\mathscr{G}_{d,t}$.

Definition 17. A ***shortsighted*** strategy σ for G is a non-empty, prefix-closed subset σ of P_G such that for any even-length $s \in \sigma$, if $sm \in P_G$ then $sm \in \sigma$ if and only if m is enabled by the last move of s. O-oriented shortsighted strategies are defined in the same way as O-oriented strategies. A ***weakly exhausting*** strategy σ is a shortsighted strategy which is O-oriented and generated by some token-reflecting bijection $f : M^P \to M^O$, and we write $\sigma = \sigma^f$ (contrast this with σ_f in Definition 5). (It is an unfortunate aspect of our terminology that shortsighted and weakly exhausting strategies are not actually strategies as defined in Section 1.)

It is straightforward to check that a weakly exhausting strategy satisfies P-exhaustion and has a response only at shortsighted positions.

In the following by a positive node in an essential net, we mean a node of positive polarity; similarly for negative node. The next lemma is a key step towards establishing a connexion between shortsighted positions and paths in $\mathscr{E}(A)$ for provable formulas A.

Lemma 18. (i) *Take a positive node t^+ of $\mathscr{E}(A)$ which is not a leaf. The subtree (of the syntactic tree of A) with root t^+ determines an* **IMLL**$^-$ *formula T such that for any leaf y there exists a path in $\mathscr{E}(A)$ from t^+ to y not crossing any axiom links if and only if y is positive and $y \in in(T)$.*

(ii) *Take a negative node t^- of $\mathscr{E}(A)$ which is not a leaf. The subtree with root t^- determines a "co-***IMLL**$^-$ *formula" $T^\perp$ such that for any leaf x there exists a path in $\mathscr{E}(A)$ from x to t^- not crossing any axiom links if and only if x is negative and $x \in in(T)$.*

(iii) *For leaves x, y there exists a path (in $\mathscr{E}(A)$) from x to y that does not cross any axiom links if and only if x is negative, y is positive and x enables y.*

Moreover the respective paths in the above are unique, if they exist.

Proof. We prove (i) and (ii) by a case-by-case induction on the depth of t. For (iii) we first notice that the outgoing edge of any positive leaf is an axiom link and so is the incoming edge of any negative leaf. Therefore x must be negative and y positive. Furthermore, the polarity of nodes in any path from x to y has to change at some point from negative to positive. The only node at which this can happen is $\otimes^-$ which means that $A = C^-[Y \multimap X]$ and by (ii) and (i) of the Lemma, x and y are initial in X and Y, respectively, so x enables y. Conversely, if x is negative, y is positive and x enables y, we can find X and Y as above and use (i) and (ii) to construct the desired path. □

Proposition 19. *Given a linearly balanced* **IMLL**$^-$ *formula A, we generate alternating sequences of moves from the associated game as follows. The first element of any sequence is some initial move (hence it is an O-move). Any sequence so of odd length is extended to sop, provided o and p are part of the same axiom link. Any even-length sequence sp extends to spo provided p enables o. There is a 1–1 correspondence between such sequences and paths in $\mathscr{E}(A)$ starting from the root and ending in leaves. Besides, maximal such sequences of moves correspond to maximal paths from the root in $\mathscr{E}(A)$.*

Proof. The first correspondence follows by repeated application of the previous lemma. Maximal sequences arise when the final P-move does not enable any O-moves. The paths generated by the previous correspondence i.e. ending in leaves, can be extended uniquely to maximal ones. Indeed, after visiting the last leaf they can only reach nodes of types $⅋^-$ and stop at the sink of $⅋^+$ (the node $\otimes^-$ would mean there is an enabled move). Alternating sequences are obtained from paths by restricting the latter to leaves. □

For provable linearly balanced formulas the sequences are exactly the shortsighted positions from the corresponding strategy. For the essential net in Fig. 2, the set of maximal such positions is $\{ccdd, ccff, ee, bb\}$. Note that the corresponding strategy also has positions which are not shortsighted e.g. *bbccff*.

Lemma 20. *Suppose* $A = C^+[X \multimap Y]$. *Then all paths in* $\mathscr{E}(A)$ *from the root to elements of* $in(Y)$ *must go through the* $⅋^+$*-node corresponding to the* $\multimap$ *in question.*

Proof. Let y^+ be an initial atom of Y. By Lemma 18 there exists a unique path from the $⅋^+$ node to y^+. Observe that it will consist of positive nodes only—the presence of a $\otimes^-$ node would mean that $y \notin in(Y)$. Because there is at most one way of entering a positive node each path from the root to y^+ will contain the above path as the final subpath. In particular, it will pass through the designated $⅋^+$ node. □

Let A be an **IMLL**$^-$ formula (which is not assumed to be linearly balanced). If there is a weakly exhausting strategy σ^f for the free game associated with A, then A can be regarded as linearly balanced with respect to the bijection f, and hence we can define the associated essential net $\mathscr{E}(A)$ *relative to* f.

Theorem 21. *Let* σ^f *be a weakly exhausting strategy for a free game associated with an* **IMLL**$^-$ *formula* A. *Then* σ^f *determines a correct essential net relative to* f, *and hence*, A *is provable by Lamarche's Theorem.*

Proof. We verify the two correctness criteria of essential nets.

Assume there is a cycle in $\mathscr{E}(A)$. It must involve leaves, as cycles cannot be formed solely by connective-nodes. Because σ satisfies P-exhaustion all leaves are reachable from the root by Proposition 19. Hence there must be an infinite path from the root containing repeated occurrences of leaves. By Proposition 19, such a path restricted to leaves gives an infinite alternating sequence of moves—call it w. Since P_A is a finite set of finite sequences, let s be the longest initial subsequence of w which is in P_A. By Lemma 8 (or rather, the claim just before it), s must end in an O-move m. But s is in σ_f which is by assumption weakly exhausting, and so, $s f(m) \in \sigma_f \subseteq P_A$, a contradiction.

Finally suppose, for a contradiction, it is possible to reach the sink of some $⅋^+$-node from the root without passing through the node itself. Let X, Y be such that $G = C^+[X \multimap Y]$ where the designated occurrence of $\multimap$ corresponds to the $⅋^+$ in question. The path from the root cannot visit any elements of $in(Y)$ by Lemma 20. However, since it reaches the sink, the last leaf preceding it must be an initial move of X. The corresponding alternating sequence is a valid position, because weak exhaustion guarantees totality for shortsighted positions. This cannot be true though, because the definition of the linear function space game would be violated: a move from X is played although no (initial) move from Y has appeared before. □

Corollary 22 (Definability). (i) *If f is a token-reflecting bijection for a free game A such that σ^f is a weakly exhausting strategy, then σ_f is a well-defined exhausting strategy.*

(ii) *For any exhausting strategy σ_f for a free game $\bigotimes \Gamma \multimap A$, there is a provable sequent $\Gamma \vdash s : A$ (with s cut-free) whose denotation in $\mathcal{G}_{d,t}$ is σ_f.*

Proof. (i) Suppose σ^f is weakly exhausting for a free game associated with a formula A. By the Theorem the associated essential net relative to f is correct, and so, we can "read back" a proof of A from it (see [15]) which is expressible as a cut-free **IMLL**$^-$ term. The denotation of that term, we know from Proposition 11, is exhausting, and induces the same linkage in A as f, and so, it must be equal to σ_f. (ii) We need only observe that σ^f is a well-defined weakly exhausting strategy contained in σ_f, then the same argument proves (ii). □

5.1. The category $\mathcal{G}_e$ of exhausting strategies

We define the category $\mathcal{G}_e$ whose objects are non-empty free games and whose maps $A \to B$ are given by exhausting strategies for $A \multimap B$. We can now show that exhausting strategies compose. Take $\mathcal{G}_{d,t}$-maps $\sigma : A \to B$ and $\tau : B \to C$ which are both given by exhausting strategies. By Corollary 22, there are (cut-free) terms s and t such that $[\![x : A \vdash s : B]\!] = \sigma$ and $[\![y : B \vdash t : C]\!] = \tau$. By the canonical interpretation, $[\![x : A : \vdash t\{s/y^B\}]\!]$ is the composition $\sigma ; \tau$ in $\mathcal{G}_{d,t}$. Now by the Cut Elimination Theorem for **IMLL**$^-$ (see [14]), there is a cut-free term r such that $x : A \vdash r = t\{s/y^B\} : C$ is **IMLL**$^-$-provable. Since $\mathcal{G}_{d,t}$ is a model of the type theory **IMLL**$^-$, we have $\sigma ; \tau = [\![x : A : \vdash r : C]\!]$ as $\mathcal{G}_{d,t}$-maps. Therefore, by Proposition 11, $\sigma ; \tau$ is an exhausting strategy, and hence, $\mathcal{G}_e$ is a subcategory of $\mathcal{G}_{d,t}$. Note also that the same interpretation of **IMLL**$^-$ makes sense in $\mathcal{G}_e$. Thus we have proved the following proposition:

Proposition 23. (i) *$\mathcal{G}_e$ is a well-defined subcategory of $\mathcal{G}_{d,t}$, and it inherits the tensor and linear implication constructions from $\mathcal{G}_{d,t}$.*

(ii) *$\mathcal{G}_e$ is a model of* **IMLL**$^-$ *using the standard interpretation (see* [14]*), even though it is not itself an autonomous category (because it has no tensor unit).* □

We quote a theorem from [13] which says that linearly balanced sequents have at most one proof.

Theorem 24 (Coherence). *For any linearly balanced* **IMLL**$^-$ *sequent $\Gamma \vdash A$, if $\Gamma \vdash s : A$ and $\Gamma \vdash t : A$ are provable, then so is $\Gamma \vdash s = t : A$.*

It follows from the theorem that for any **IMLL**$^-$ sequent $\Gamma \vdash A$ (which is not necessarily linearly balanced), and for any provable $\Gamma \vdash s : A$ and $\Gamma \vdash t : A$, we have $\Gamma \vdash s = t : A$ is provable if and only if s and t induce the same linkage on the sequent. Hence by Corollary 22, we have the following full completeness theorem:

Theorem 25 (Full completeness). *$\mathcal{G}_e$ is fully complete for* **IMLL**$^-$ *in the sense that for any sequent $\Gamma \vdash A$, for every exhausting strategy σ_f for the free game $\bigotimes \Gamma \multimap A$, there is some typing judgement $\Gamma \vdash s : A$ s such that for any t, the denotation of $\Gamma \vdash t : A$ is equal to σ_f if and only if $\Gamma \vdash s = t : A$ is provable.* □

Remark 26. (i) The empty game $\emptyset$ is not an object of $\mathcal{G}_e$ and this is for a good reason: there can be no exhausting strategy for the game $((A \multimap \emptyset) \otimes A) \multimap \emptyset$.

(ii) In a draft of this paper, we speculated that exhausting strategies could be used to construct a reasonably efficient "tautology checker" for IMLL i.e. to decide if a given IMLL formula is provable. We have not made much progress in that direction, instead we have obtained a linear-time tautology checker for (linearly balanced) IMLL [18] by using a fast algorithm for checking the second correctness criterion of essential nets. The algorithm can be extended to a linear-time tautology checker for (linearly balanced) MLL.

6. Connexions with fair games

For the purpose of comparison, we take a slight detour and briefly sketch another fully complete model for **IMLL**$^-$ based on the notion of *fair games* [10]. Fairness so restricts positions on linear function space games $A \multimap B$ that players can reach a maximal position in B only if a maximal position has already been reached in A in the same play so far.

Definition 27. A ***fair game*** G is a triple $\langle M_G, \lambda_G, F_G \rangle$ where M_G is a non-empty set of an even number of moves, λ_G is defined as before, and F_G is an anti-chain (w.r.t. prefix ordering) of even-length alternating sequences of moves, each beginning with an O-move. We require F_G to contain at least a non-void sequence; members of F_G are called ***fair positions***.

Every fair game G can be viewed as a game (in the sense of Definition 1) by taking P_G to be the set of all prefixes of the fair positions of G i.e. by regarding the fair positions as maximal positions. The effect is that at any position p in a fair game, a player can make a particular move m if and only if pm can be extended to some fair position. Tensor and linear function space constructions for fair games are defined in terms of fair positions.

The ***Tensor game*** $A \otimes B$ is defined as follows:

$$M_{A\otimes B} = M_A + M_B,$$

$$\lambda_{A\otimes B} = [\lambda_A, \lambda_B]$$

and $s \in F_{A\otimes B}$ if and only if s is a finite alternating sequence of moves from $M_A + M_B$, beginning with an O-move, such that $s \upharpoonright A \in F_A$ and $s \upharpoonright B \in F_B$. It is a consequence of the definition that $s \in F_{A\otimes B}$ satisfies the O-switching condition.

The ***Linear function space game*** $A \multimap B$ is defined as follows:

$$M_{A \multimap B} = M_A + M_B,$$

$$\lambda_{A \multimap B} = [\overline{\lambda_A}, \lambda_B]$$

and $s \in F_{A \multimap B}$ if and only if s is a finite alternating sequence of moves from $M_A + M_B$, beginning with an O-move, such that $s \upharpoonright A \in F_A$ and $s \upharpoonright B \in F_B$. It follows from the definition that $s \in F_{A \multimap B}$ satisfies the P-switching condition.

For each a from a fixed universe of tokens, we define the ***basic fair game*** B_a as follows:

$$B_a = \langle \{a_{\mathsf{q}}, a_{\mathsf{a}}\}, \{(a_{\mathsf{q}}, O), (a_{\mathsf{a}}, P)\}, \{a_{\mathsf{q}} a_{\mathsf{a}}\} \rangle.$$

We can think of the two moves a_{q} and a_{a} as a pair of matching question and answer (hence the subscripts). We shall be concerned with fair games that are freely generated from the basic games using the tensor and linear function space constructions. Call these games ***free fair games***. It is easy to see that if G is one such, then there is a *perfect matching*, between the question-moves and the answer-moves of G, which is determined by the free construction. That is to say, let Q_G and A_G, respectively, be the subsets of question-moves and answer-moves of the free fair game G, then G comes equipped with a bijection $\mathsf{ans} : \mathsf{Q}_G \to \mathsf{A}_G$ which is "lifted" from the respective matchings $a_{\mathsf{q}} \mapsto a_{\mathsf{a}}$ of the component basic games.

Definition 28. A strategy σ for a free fair game G is ***linking*** if σ is O-oriented and generated by some bijection f such that for question-moves q and q' of G

$$f(q) = q' \Leftrightarrow f(\mathsf{a}ns(q')) = \mathsf{a}ns(q).$$

We can extract the following result from [10].

Theorem 29 (Full completeness). *Linking strategies and free fair games are a fully complete model of* **IMLL**$^-$.

It is worth relating the fair games model with the one based on exhausting strategies: every position of the latter can be extended to a position of the former. This is one reason why we decided to present our model from scratch rather than extracting it from the (more general) fair games model. The fairness condition also has the consequence that both players must explore the fair positions in order to make the next move.

7. The intermediate system IMLL$^\flat$

Our ultimate goal is to give a fully complete interpretation of proofs of IMLL with the tensor unit. So far we have not yet considered the tensor unit $\top$, for which

(id-atom) $x : a \vdash x^a : a$ (a is atomic and non-unit)

(exch) $\dfrac{\Gamma, x : A, y : B, \Delta \vdash s : C}{\Gamma, y : B, x : A, \Delta \vdash s : C}$

(cut) $\Gamma, x : A \vdash s : B \qquad \Delta \vdash t : A\Gamma, \Delta \vdash s\{t/x^A\} : B$

($\top$-l) $\dfrac{\Gamma \vdash s : A}{\Gamma, x : \top \vdash \langle x^\top/*\rangle s : A}$

($\top$-r) $\vdash * : \top$

($\otimes$-l) $\dfrac{x : A, y : B, \Gamma \vdash s : C}{z : A \otimes B, \Gamma \vdash \langle z^{A \otimes B}/x^A \otimes y^B\rangle s : C}$

($\otimes$-r) $\dfrac{\Gamma \vdash s : A \qquad \Delta \vdash t : B}{\Gamma, \Delta \vdash s \otimes t : A \otimes B}$

($\multimap$-l) $\dfrac{\Gamma \vdash s : A \quad y : B, \Delta \vdash t : C}{z : A \multimap B, \Gamma, \Delta \vdash \langle z^{A \multimap B}, s/y^B\rangle t : C}$

($\multimap$-r) $\dfrac{\Gamma, x : A \vdash s : B}{\Gamma \vdash \lambda x^A.\ s : A \multimap B}$

Fig. 3. Rules defining the valid typing judgements of **IMLL**.

the type theory **IMLL** has the following typing rules

$$(\top\text{-r}) \quad \frac{}{\vdash * : \top} \qquad (\top\text{-l}) \quad \frac{\Gamma \vdash s : A}{\Gamma x\!:\ \top \vdash \langle x^\top/*\rangle s : A}$$

and an additional equality axiom

$$(\top) \quad \Gamma \vdash (\langle x^\top/*\rangle s)\{*/x\} = s : A.$$

In contrast to the tensor-let and lambda-let, note that the ***unit-let*** construct $\langle x^\top/*\rangle-$ does not bind any variables. Note also that the definition of context C has an additional rule $C ::= \langle x^\top/*\rangle C$; and since π now ranges over an extra let-construct $\langle x^\top/*\rangle-$, the axiom scheme (π-**cong**) has a new instantiation of the form

$$\Theta \vdash C[\langle x^\top/*\rangle t] \sim \langle x^\top/*\rangle C[t] : A.$$

For ease of reference we gather all the rules that define **IMLL** in Fig. 3.

Remark 30. The type theory **IMLL** is a slight variant of the Autonomous Type Theory as introduced in [14]. The difference is that the type A in the identity axiom

$$x : A \vdash x^A : A$$

here is restricted to an atomic and non-unit type. Consequently the three η-axioms (for the equality judgements) are not necessary and have been omitted from **IMLL**.

7.1. *The intermediate type theory* $\mathbf{IMLL}^\flat$

We shall first consider an intermediate type theory $\mathbf{IMLL}^\flat$. The virtue of $\mathbf{IMLL}^\flat$ is that its syntax is comparatively easy to analyse, and yet it captures all **IMLL** terms up to provable equality in the sense of Lemma 33. The typing rules defining $\mathbf{IMLL}^\flat$ are those of $\mathbf{IMLL}^-$ (with the proviso that the type a in the typing axiom (id-atom) ranges over non-$\top$ atoms) augmented by

$$(\top\text{-}\mathbf{l}\text{-}\mathbf{v}ar)\ \frac{}{x_1 : \top, \ldots, x_n : \top, x : a \vdash \langle x_1/*\rangle \cdots \langle x_n/*\rangle x^a : a},$$

$$(\top\text{-}\mathbf{l}\text{-}\mathbf{r})\ \frac{}{x_1 : \top, \ldots, x_n : \top \vdash \langle x_1/*\rangle \cdots \langle x_n/*\rangle * : \top},$$

where $n \geqslant 0$. Note that the first rule subsumes the rule (id-atom); note also that $x : \top \vdash x : \top$ is not provable but $x : \top \vdash \langle/*\rangle : *\top$ is.

The order in which the unit-let constructs $\langle x^\top/*\rangle-$ are introduced is considered irrelevant in $\mathbf{IMLL}^\flat$ so that terms are required to satisfy the following π-congruence axiom (Fig. 4):

$$\Gamma \vdash C[\langle x_1/*\rangle \cdots \langle x_n/*\rangle t] \sim C[\langle x_{p(1)}/*\rangle \cdots \langle x_{p(n)}/*\rangle t] : A,$$

where p is a permutation of $\{1, \ldots, n\}$, t is either a variable or the constant $*$, and where C is defined by recursion over the same rules that define the one-holed contexts of $\mathbf{IMLL}^-$ (note though that the s and t in those rules now range over $\mathbf{IMLL}^\flat$ terms). There are three additional axioms in $\mathbf{IMLL}^\flat$ for the equality judgements: $n, k \geqslant 0$

$$(\langle x_1/*\rangle \cdots \langle x_n/*\rangle x^a)\{\langle y_1/*\rangle \cdots \langle y_k/*\rangle z^a / x^a\}$$
$$= \langle x_1/*\rangle \cdots \langle x_n/*\rangle \langle y_1/*\rangle \cdots \langle y_k/*\rangle z^a : a$$

$$(\langle x_1/*\rangle \cdots \langle x_n/*\rangle *)\{\langle y_1/*\rangle \cdots \langle y_k/*\rangle * / x_n^\top\}$$
$$= \langle x_1/*\rangle \cdots \langle x_{n-1}/*\rangle \langle y_1/*\rangle \cdots \langle y_k/*\rangle * : \top$$

$$(\langle x_1/*\rangle \cdots \langle x_n/*\rangle z^a)\{\langle y_1/*\rangle \cdots \langle y_k/*\rangle * / x_n^\top\}$$
$$= \langle x_1/*\rangle \cdots \langle x_{n-1}/*\rangle \langle y_1/*\rangle \cdots \langle y_k/*\rangle z^a : a$$

(refl) $\Gamma \vdash s = s : A$

(symm) $$\frac{\Gamma \vdash s = t : A}{\Gamma \vdash t = s : A}$$

(trans) $$\frac{\Gamma \vdash s = t : A \qquad \Gamma \vdash t = u : A}{\Gamma \vdash s = u : A}$$

(context) $$\frac{\Gamma \vdash s = t : A \qquad \Gamma, \Delta \vdash C[s] : B}{\Gamma, \Delta \vdash C[s] = C[t] : B}$$

(cong) $$\frac{\Gamma \vdash s \sim t : A}{\Gamma \vdash s = t : A}$$

(id) $\Gamma \vdash x^a\{s/x^a\} = s : a$ (a is a non-$\top$ atomic type)

($\top$) $\Gamma \vdash (\langle x^\top/*\rangle s)\{*\ /x\} = s : A$

($\otimes$) $\Gamma \vdash (\langle z^{A \otimes B}/x \otimes y\rangle s)\{u \otimes v/z\} = (s\{u/x^A\})\{v/y^B\} : C$

($\multimap$) $\Gamma \vdash (\langle z^{A \multimap B}, s/y^B\rangle t)\{\lambda x^A.u/z\} = t\{u\{s/x^A\}/y^B\} : C$

Fig. 4. Rules defining the valid equality judgements of **IMLL**.

(we omit the typing context in the above). It is straightforward to prove a cut-elimination result using a rewrite system along the same lines as that for **IMLL**$^-$.

Proposition 31 (Cut elimination). *If $\Gamma \vdash s : A$ is provable in* **IMLL**$^\flat$ *then there is a cut-free s' such that $\Gamma \vdash s = s' : A$ is provable in* **IMLL**$^\flat$.

One can think of the ($\top$-l-_) rules as introducing links between the occurrences of unit on the left and the atomic type on the right (of the turnstile). In **IMLL**$^\flat$ each negative occurrence of an atomic type is an end of exactly one link, whereas each positive occurrence may be an end of more than one links (the ($\top$-l) rules with $n \geqslant 1$ create such) or it may not be linked at all (($\top$-l-r) where $n = 0$). These links play a crucial role in the proof theory. Consider, for example, the following **IMLL**$^\flat$ terms (we distinguish the two occurrences of $\top$ as $\top_1$ and $\top_2$)

(i) $x_1 : \top_1, x_2 : \top_2, y : b, z : c \vdash (\langle x_1/*\rangle\langle x_2\, y/*\rangle) \otimes z : b \otimes c$,

(ii) $x_1 : \top_1, x_2 : \top_2, y : b, z : c \vdash (\langle x_2/*\rangle\langle x_1/*\rangle\, y) \otimes z : b \otimes c$,

(iii) $x_1 : \top_1, x_2 : \top_2, y : b, z : c \vdash \langle x_1/*\rangle\, y \otimes \langle x_2/*\rangle z : b \otimes c$

(**refl**) $\Gamma \vdash s \sim s : A$

(**symm**) $\dfrac{\Gamma \vdash s \sim t : A}{\Gamma \vdash t \sim s : A}$

(**trans**) $\dfrac{\Gamma \vdash s \sim t : A \quad \Gamma \vdash t \sim u : A}{\Gamma \vdash s \sim u : A}$

(**context**) $\dfrac{\Gamma \vdash s \sim t : A \quad \Gamma, \Delta \vdash C[s] : B}{\Gamma, \Delta \vdash C[s] \sim C[t] : B}$

(π-**cong**) $\Gamma \vdash \pi C[t] \sim C[\pi t] : A$

(σ-**cong**) $\Gamma \vdash C[t]\sigma \sim C[t\sigma] : A$

Fig. 5. Rules defining the valid congruence judgements of **IMLL**. The last two axioms are required to satisfy the *strong typability* side condition: the expressions on both sides of $\sim$ be typable of the same declared type.

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(i) and (ii) induce the same linkage ($\top_1$ and $\top_2$ are both linked to b) which is different from that induced by (iii) ($\top_2$ is linked to c).

As the next lemma shows, $\mathbf{IMLL}^\flat$ terms of the same type which induce the same linkage between positive and negative occurrences of atoms are equal.

Lemma 32. *In* $\mathbf{IMLL}^\flat$, *if* $\Gamma \vdash t_1 : A$ *and* $\Gamma \vdash t_2 : A$ *are provable and determine the same linkage then* $\Gamma \vdash t_1 = t_2 : A$ *is provable.*

Proof. W.l.o.g. we can assume that t_1 and t_2 are cut-tree (i.e. have no explicit substitution subterm). We use induction on the number of occurrences of $\top$ in $\Gamma \vdash A$. If there is none, we are done. Otherwise, there are two cases. First suppose there is at least one positive occurrence. There are two subcases:

- If no other unit connects to it by a link, then for some C[-], both t_i have one of the following shapes in common:

  $$C_i[* \otimes t], \quad C_i[t \otimes *] \quad \text{or} \quad C_i[\langle x^{\top \multimap B}, */y^B \rangle t],$$

  where the occurrence of $*$ corresponds to that occurrence of $\top$ in $\Gamma \vdash A$ and so, it is uniquely determined in each t_i. Now construct t_i' by "erasing" $* \otimes -$, $- \otimes *$, $\langle x^{\top \multimap B}, */y^B \rangle -$, respectively. In the last case the unique occurrence of y in t is replaced by x whose type is changed to B. It remains to change all type annotations associated with $\top \multimap B$ to B in order to get a typable term. By the induction hypothesis t_1' and t_2' are equivalent. The passage from one to another can be adapted to one for t_1 and t_2.

- Otherwise suppose there is another occurrence of $\top$ linked to it. Then we have $t_i = C_i[\langle x^{\top}/*\rangle *]$ (up to $\sim$) where $*$ and $\langle x^{\top}/*\rangle -$ are the respective constructs corresponding to the two occurrences. Let $t_i' = C_i[y]$ where y is of a fresh atomic type Y and appeal to the induction hypothesis.

Secondly, there is no positive occurrence of $\top$ in $\Gamma \vdash A$ but there is a negative one. Then we have $t_i = C_i[\langle x_{i1}^{\top}/*\rangle \ldots \langle x_{ik_i}^{\top}/*\rangle \langle x^{\top}/*\rangle y]$, where x, y correspond in both terms to the respective occurrences of the unit and the matched atomic type. Take $t_i' = C_i[\langle x_{i1}^{\top}/*\rangle \ldots \langle x_{ik_i}^{\top}/*\rangle y \otimes x^a]$ where $x's$ type is a fresh atom so as to get a term whose type has fewer occurrences of the unit. Again, it is necessary to replace any occurrences of $\top$ related with x to a if t_i' is to be a typable term. We then conclude by appealing to the induction hypothesis. □

Lemma 33. *For any* **IMLL**-*provable* $\Gamma \vdash t : A$, *there is a* $\mathbf{IMLL}^{\flat}$-*term* t' *such that* $\Gamma \vdash t \sim t' : A$ *is* **IMLL**-*provable.*

Proof. Easy structural induction. Consider the expressions that can appear immediately to the right of the occurrence of $\langle z^{\top}/*\rangle -$. □

The lemma shows that every $=$-equivalence class of terms in **IMLL** has some representative in $\mathbf{IMLL}^{\flat}$. Consequently we can derive the following result.

Corollary 34 (Coherence 1). *If* $\Gamma \vdash A$ *is linearly balanced and contains only positive occurrences of the unit, then there is at most one* $=$*-equivalence class of terms inhabiting* $\Gamma \vdash A$ *in* $\mathbf{IMLL}^{\flat}$ (**IMLL**).

7.2. Essential nets with units

Having observed that $\mathbf{IMLL}^{\flat}$ is based on links between occurrences of atoms of opposite polarities we extend the definition of essential nets to take account of the non-standard links between positive occurrences of atoms and negative occurrences of the tensor unit. The right introduction rule of the tensor unit provides a new node, whereas introductions on the left in addition to a new node bring a link from a positive occurrence of an atom to the negative occurrence of the unit. The links leading from negative occurrences of units will be called ***unit links***. An easy induction shows that all derivations yield essential nets satisfying the two correctness criteria. We will show that they are enough to ensure provability. For an example see Fig. 6.

Theorem 35 (Sequentialization). *A correct essential net for* $\mathbf{IMLL}^{\flat}$ *is sequentializable. I.e. for any correct essential net for a formula* Γ *there is a* $\mathbf{IMLL}^{\flat}$ *derivation for* Γ.

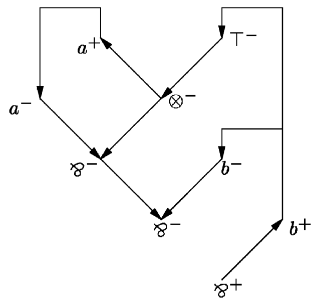

Fig. 6. The essential net of $(a \otimes (a \multimap G_*)) \otimes b \multimap b$.

Proof. We use induction on the number of negative occurrences of the unit in the formula Γ. If there are none, the unit may occur positively in the following contexts $C^+[\top \otimes T]$, $C^+[T \otimes \top]$ or $C^-[\top \multimap T]$. In all cases, the occurrence of the unit ends all paths visiting it. Thus, if we erase the units and associated nodes, we get a correct essential net for **IMLL**$^-$ which by Theorem 12 is sequentializable. That sequentialization defines an **IMLL**$^\flat$ derivation by introducing the units as soon as T is introduced.

Suppose there are $n+1$ negative occurrences of the unit. Let us pick an arbitrary one $\top$ and let x be the source of the associated unit link so that $\Gamma = C[\top^-, x^+]$. Consider now a new net for $C[\top, x \otimes \top_1]$ in which the unit link from x now has $\top_1$ as its source and the other links remain as before. If we replace the new unit link with a non-unit axiom link joining a^+ and a^- for some fresh atom a, we can appeal to the induction hypothesis and get a sequentialization of $C[a, x \otimes a]$. That sequentialization has to be modified to yield a proper **IMLL**$^\flat$ derivation for the original sequent. This is quite easy: omit the introduction of the distinguished non-unit axiom link, and introduce the unit link when the axiom link for x is introduced. □

8. Joker moves and conditionally exhausting strategies

We can now consider the game interpretation of the tensor unit. We define the atomic game

$$G_* = \langle \{*\}, \{(*, O)\}, \{\varepsilon, *\} \rangle,$$

where $*$ here (by abuse of notation) is a new distinguished token (which is assumed not to be an element of $\mathscr{U}$), for which special game rules as follows apply:
(j1) P may play $*$ in response to any O-move.
(j2) When O plays $*$, P is not obliged to respond, but if he does, he must do so with $*$.

Note that $*$ is available as a P-move, and so (j1) is applicable, in a game which has G_* as a subgame "in a negative position"; similarly $*$ is available as an O-move in a game which has G_* as a subgame "in a positive position". We regard $*$ as a kind of joker move (for P) as these rules are biased towards P. We define a ***joker move*** to be a move whose token is $*$, and refer to the game G_* as the *Joker Game*. By interpreting the tensor unit as the Joker Game, the following types:

$$\top \multimap \top, \quad \top \otimes \top, \quad \top$$

are distinguished (they are all identified by the semantics in [2] which interprets the unit as the empty game). Also games of the shape $A \multimap G_*$ can have non-trivial sets of positions, in contrast to $A \multimap \emptyset$.

Remark 36. Another way to think of the $*$-moves is as O's "surrender moves": whenever O plays $*$, he surrenders, and P wins by default.

We extend exhausting strategies to interpret joker moves in accord with the above rules. This leads to a new class of strategies.

Definition 37. A strategy σ for A is ***conditionally exhausting*** if it is O-oriented and generated by some (total) function $f : M^P \to M^O$ (i.e. $\sigma = \sigma_f$) such that
(i) f is bijective and token-reflecting when restricted to non-joker moves,
(ii) if $f(x)$ is a joker move then so is x.
We refer to f as the ***linkage*** (between atomic types of A) associated with σ.

The reader might wish to check that conditionally exhausting strategies satisfy the P-exhaustion condition. Note that it is possible for a conditionally exhausting strategy to be
(i) non-deterministic, since we may have $f(j) = f(m)$ for some joker move j and non-joker move m; e.g. the canonical strategy for $a \otimes G_* \multimap a$
(ii) partial, since there may be some O-move, which must be a joker move, that is not in the image of f; e.g. the canonical strategy for $a \multimap G_* \otimes a$.
But at odd-length positions ending with a non-joker O-move and where the joker move is not available to P, *local* determinacy and totality of the strategy are guaranteed. For this reason, we call the strategies *conditionally* exhausting. It follows from the definition that if the game has no joker move then the notions of exhausting and conditionally exhausting coincide.

There is a correspondence between the linkage f which defines a conditionally exhausting strategy, and the linkage given by the $\mathbf{IMLL}^\flat$ unit rules, which is the crux of our main result. In fact we shall see that every term (in context) in $\mathbf{IMLL}^\flat$ is interpreted by a conditionally exhausting strategy.

Example 38. (i) There is a conditionally exhausting strategy for the game

$$a \otimes (a \multimap G_*) \otimes b \multimap b$$

given by the function $b \mapsto b, * \mapsto b, a \mapsto a$. The corresponding essential net is shown in Fig. 6.

(ii) There are three conditionally exhausting strategies for $a\otimes(a\multimap G_{*_1})\otimes b\otimes c\otimes(c\multimap G_{*_2})\multimap b$; their respective generating functions, restricted to joker moves, are:
(a) $f_1 : *_1 \mapsto b, *_2 \mapsto a$
(b) $f_2 : *_1 \mapsto b, *_2 \mapsto b$
(c) $f_3 : *_1 \mapsto c, *_2 \mapsto b$.

(iii) The game $\top(m,n) = \underbrace{G_* \otimes \cdots \otimes G_*}_{m} \multimap \underbrace{G_* \otimes \cdots \otimes G_*}_{n}$ admits n^m conditionally exhausting strategies, whereas

$$((a \multimap G_*) \multimap G_*) \multimap ((a \multimap G_*) \multimap G_*)$$

admits only one.

(iv) There are two conditionally exhausting strategies for the game:

$$((a \multimap G_{*_1}) \multimap G_{*_2}) \otimes ((b \multimap G_{*_3}) \multimap G_{*_4}) \multimap (((a \otimes b) \multimap G_{*_5}) \multimap G_{*_6})$$

defined by
(a) $*_2 \mapsto *_3,\ *_4 \mapsto *_6, *_5 \mapsto *_1$
(b) $*_2 \mapsto *_6,\ *_4 \mapsto *_1, *_5 \mapsto *_3$.
This example is due to Barry Jay.

8.1. Interpretation of **IMLL**$^\flat$ in $\mathcal{G}$, and the subcategory $\mathcal{G}_{ce}$

Recall the category $\mathcal{G}$ of free games and (non-deterministic and partial) strategies introduced in Definition 1. From now on, by $\mathcal{G}$ we shall mean the same category but the object-generators now include G_*. We give an interpretation of **IMLL**$^\flat$ in the category $\mathcal{G}$. The denotation of a **IMLL**$^\flat$ sequent as a $\mathcal{G}$-map is written $[\![\Gamma \vdash s : A]\!]^\flat$, and its definition for the **IMLL**$^-$-fragment is exactly the same as the canonical interpretation of **IMLL**$^-$ in any autonomous category (considered in Section 3 and defined in [14]). For the unit part, we define $[\![\top]\!]^\flat = G_*$; for $n \geqslant 0$ and $a \in \mathcal{U}$

(i) we define $[\![x_1 : \top, \ldots, x_n : \top,\ x : a \vdash \langle x_1/*\rangle \cdots \langle x_n/*\rangle x : a]\!]^\flat$ to be the non-deterministic, total strategy σ_f for the game $(\otimes_{i=1}^n G_* \otimes a) \multimap a$ where $f : *_1 \mapsto a, \ldots, *_n \mapsto a, a \mapsto a$, and
(ii) we define $[\![x_1 : \top, \ldots,\ x_n : \top \vdash \langle x_1/*\rangle \cdots \langle x_n/*\rangle * : \top]\!]^\flat$ be the non-deterministic strategy σ_f for the game $(\otimes_{i=1}^n G_*) \multimap G_*$ given by $f : *_1 \mapsto *, \ldots, *_n \mapsto *$; note that σ_f is total if and only if $n > 0$.

We take $\otimes_{i=1}^0 G_* = \emptyset$ so that $[\![\ \vdash * : \top]\!]^\flat$ is the undefined strategy for the game G_*.

Proposition 39. *If $\Gamma \vdash s : A$ is provable and s is cut-free in* **IMLL**$^\flat$ *then $[\![\Gamma \vdash s : A]\!]^\flat$ is a conditionally exhausting strategy.*

Proof. Similar to the proof of Proposition 11. □

As expected, every conditionally exhausting strategy is the denotation of some **IMLL**$^\flat$ term. Weakly conditionally exhausting strategies are defined in a similar way to weakly exhausting strategies and we also denote them by σ^f. Observe that the correspondence between shortsighted sequences and paths in the essential net is analogous to that described in Section 3. That way we get an immediate proof of an analogue of the first part of Corollary 22 for conditionally exhausting strategies:

Theorem 40. *If σ^f is a weakly conditionally exhausting strategy for G, then f defines a conditionally exhausting strategy σ_f for G and G is* $\mathbf{IMLL}^\flat$*-provable.*

Proof (**Sketch**). We shall confuse free games with types in the following. We prove this by induction on the number of negative occurrences of $\top$ in the type $\Phi = \otimes\Gamma \multimap A$. If there is none, let Φ^- be the type obtained from Φ by erasing all positive occurrences of $\top$ in Φ. Now Φ^- is an $\mathbf{IMLL}^-$ game, so the linkage f determines an exhausting strategy for Φ^- which is the denotation of a term s^- by Corollary 22; from s^- we can obtain s where $[\![\Gamma \vdash s : A]\!]^\flat = \sigma_f$. Now suppose there is a negative occurrence of $\top$ in Φ. Create a new link between two atoms by replacing the occurrence of $\top$ itself by one fresh atom and adding, with $\otimes$, an occurrence of the same atom to the atom at the other end of the link to get Φ'. The number of negative occurrences of $\top$ in Φ' is strictly smaller than that in Φ. The resultant linkage, call it f', determines a conditionally exhausting strategy for the "smaller" game Φ'; by the induction hypothesis, the strategy $\sigma_{f'}$ is the denotation of some $\mathbf{IMLL}^\flat$-term s^-. Again from s^- we can obtain s such that $[\![\Gamma \vdash s : A]\!]^\flat = \sigma_f$ as required. $\square$

We define a new category $\mathcal{G}_{ce}$ whose objects are (non-empty) free games generated from atomic games $G_*, G_a, G_b, \ldots$ by the tensor and linear implication, and whose maps are given by conditionally exhausting strategies. Since the interpretation of $\mathbf{IMLL}^\flat$ in $\mathcal{G}$ is sound (we easily check that the new $\mathbf{IMLL}^\flat$-axioms are validated; the other axioms from $\mathbf{IMLL}^-$ are valid because $\mathcal{G}$ is autonomous), and since we have Cut Elimination for $\mathbf{IMLL}^\flat$ (Proposition 31), Proposition 39 and the Definability Theorem 40, we can prove the compositionality of conditionally exhausting strategies by exactly the same reasoning as the compositionality of exhausting strategies. Note that the composition is the standard one inherited from $\mathcal{G}$. Thus we can conclude:

Theorem 41 (Full completeness). (i) *The category $\mathcal{G}_{ce}$ is a well-defined subcategory of $\mathcal{G}$ which respects the tensor and linear implication.*
(ii) *$\mathcal{G}_{ce}$ is a fully complete model for* $\mathbf{IMLL}^\flat$.

9. A fully complete game model for IMLL

Though the category $\mathcal{G}_{ce}$ is equipped with a tensor bifunctor and linear implication, it is not autonomous because it lacks a tensor unit (the obvious candidate G_* is not the unit since there are n^m conditionally exhausting strategies for the game $\top(m,n)$ in Example 38(iii)). In this section, we aim to turn $\mathcal{G}_{ce}$ into an autonomous category by quotienting its homsets by an appropriate equivalence relation.

Let $\boldsymbol{J}$ be the universe of joker moves. For conditionally exhausting strategies $\sigma_1 = \sigma_{f_1}$ and $\sigma_2 = \sigma_{f_2}$ for the same game G, we define the (reflexive and symmetric) relation $\sim$ as follows:

$$\sigma_{f_1} \sim \sigma_{f_2} \Leftrightarrow \exists j \in \boldsymbol{J}.\forall x \in M_G.f_1(x) \neq f_2(x) \Rightarrow x = j.$$

In words $\sigma_{f_1} \sim \sigma_{f_2}$ means that the two generating functions f_1 and f_2 agree everywhere except at a specific joker move j. For instance, in Example 38(ii), we have $\sigma_{f_1} \sim \sigma_{f_2}$ and $\sigma_{f_2} \sim \sigma_{f_3}$ but $\sigma_{f_1} \not\sim \sigma_{f_3}$.

Definition 42. We set the equivalence relation $\approx$ to be the transitive closure of $\sim$, and define $\mathscr{G}_a$ to be the category whose objects are the $\mathscr{G}_{ce}$-objects and whose maps $A \to B$ are given by $\approx$-equivalence classes of conditionally exhausting strategies for $A \multimap B$.

Our aim is to show that this relation characterizes the equivalence of essential nets needed to ensure their compatibility with the autonomous theory. The idea is that two essential nets with units should be deemed equivalent if and only if it is possible to transform one to another by changing the positive end of unit links, one at a time, so that the intermediate linkages also define essential nets.

Theorem 43. *$\mathscr{G}_a$ is a well-defined autonomous category.*

Proof. We prove that if $\sigma_{f_1} \sim \sigma_{f_2}$ then
(i) $\sigma_{f_1} \otimes \tau \sim \sigma_{f_2} \otimes \tau$,
(ii) $\tau \otimes \sigma_{f_1} \sim \tau \otimes \sigma_{f_2}$,
(iii) $\sigma_{f_1}; \tau \approx \sigma_{f_2}; \tau$,
(iv) $\tau; \sigma_{f_1} \approx \tau; \sigma_{f_2}$.
The first two are easy. The third is analogous to the fourth, so we just prove the last one. Because the composition of conditionally exhausting strategies is a conditionally exhausting strategy, the composites $\tau; \sigma_{f_1}$ and $\tau; \sigma_{f_2}$, where $\tau : A \to B$ and $\sigma_{f_i} : B \to C$, are generated by functions g_1 and g_2, respectively. Let j be the joker move in $M_{B \multimap C}$ that witnesses $\sigma_{f_1} \sim \sigma_{f_2}$. There are two cases. First suppose j is in the C-component. Then for any $x \in M_{A \multimap C}$, $g_1(x) = g_2(x)$ iff $x \neq j$ by the definition of composition, and so, $\sigma_{g_1} \sim \sigma_{g_2}$ and we are done. Otherwise suppose j is in the B-component. Then there is a set $\star = \{*_1, \ldots, *_n\} \subseteq M_{A \multimap C}$ of joker moves (of the game $A \multimap C$) such that:
(a) for $i = 1, 2$, for $j, k \in \star$, $g_i(j) = g_i(k)$, and
(b) if $g_1(j) \neq g_2(j)$ then $j \in \star$.
We need the following lemma in order to prove $\sigma_{g_1} \approx \sigma_{g_2}$.

Lemma 44. *For each $0 \leqslant i \leqslant n$, h_i as defined by*

$$h_i(j) = \begin{cases} g_1(j)(= g_2(j)), & j \notin \bigstar, \\ g_1(j), & j \in \{*_1, \ldots, *_i\}, \\ g_2(j), & j \in \{*_{i+1}, \ldots, *_n\}. \end{cases}$$

induces a conditionally exhausting strategy.

Proof. We prove that h_i defines a correct essential net. First we show that all paths in the net come from the essential nets corresponding to g_1 or g_2. Because g_1 and g_2 define correct nets, so will h_i.

Suppose there is a path in the net defined by h_i that is not a path from the net corresponding to g_1 or g_2 i.e. one that crosses an axiom link between $g_1(j)$ and $\top_j$ for $j \leqslant i$ and an axiom link between $g_2(j)$ and $\top_j$ for $j > i$. This means that there exists a path from $\top_j$ to $g_k(j)$ for either some $j \leqslant i$ and $k = 2$, or $j > i$ and $k = 1$ (depending on the order in which the two axiom links occur in the path) that is a path in the nets induced both by g_2 and g_1. But the nets have axiom links between $g_2(j)$ and $\top_j$ and between $g_1(j)$ and $\top_j$ respectively, so the existence of such a path would contradict acyclicity of one of them. □

Because $h_0 = g_2$, $h_n = g_1$ and $\sigma_{h_i} \sim \sigma_{h_{i+1}}$ for $0 \leqslant i < n$, we have $\sigma_{g_1} \approx \sigma_{g_2}$. The category $\mathscr{G}_{ce}$ already features tensor, linear function space and the canonical morphisms $\alpha_{A,B,C}$ and $\sigma_{A,B}$ making the corresponding diagrams commute. By (i)–(iv) this structure will be preserved in $\mathscr{G}_a$. Hence, it suffices to define the maps l_A. and r_A so that the remaining diagrams commute.

Take any game A and let m be an opening move (in fact any O-move). We define $l_A : \top \otimes A \to A$ as the $\approx$-equivalence class of σ_{f_m} where $f_m : * \mapsto m$ but, when restricted to the non-joker moves, generates the identity map $A \to A$. (Of course the choice of m is immaterial.) We define r_A similarly. We invite the reader to check that l_A and r_A are isos natural in A, and that all the required diagrams commute. □

Our task now is to prove that $\mathscr{G}_a$ is fully complete for **IMLL**. The "gap" between **IMLL**$^\flat$ and **IMLL** is the axiom (π-**cong**) in all its generality when π is instantiated to the unit-let $\langle x^\top/*\rangle$–

$$\Gamma \vdash C[\langle x^\top/*\rangle t] \sim \langle x^\top/*\rangle C[t] : A$$

subject, of course, to the strong typability side condition. Because $\mathscr{G}_a$ is autonomous, it is a model of **IMLL**, and so, if $\Gamma \vdash s = t : A$ is provable in **IMLL**$^\flat$ we must have $[\![\Gamma \vdash s : A]\!]^\flat \approx [\![\Gamma \vdash t : A]\!]^\flat$. It remains to show that if two conditionally exhausting strategies for the same game are $\approx$-related, then the respective **IMLL**$^\flat$ terms determined by the strategies (by Theorem 40) are equal in **IMLL**. The following proposition allows us to infer that.

Proposition 45. *Suppose we have provable* **IMLL**$^\flat$ *sequents* $\Gamma \vdash s : A$ *and* $\Gamma \vdash t : A$ *such that*

$$[\![\Gamma \vdash s : A]\!]^\flat \sim [\![\Gamma \vdash t : A]\!]^\flat,$$

then $\Gamma \vdash s = t : A$ *is provable in* **IMLL**.

Proof. By assumption $[\![\vdash \boldsymbol{\lambda} x.s : \otimes \Gamma \multimap A]\!]^\flat$ and $[\![\vdash \boldsymbol{\lambda} x.t : \otimes \Gamma \multimap A]\!]^\flat$ conditionally exhausting strategies corresponding to linkages that differ only by one link whose target is the same negative occurrence of the joker move. Set $\otimes\Gamma \multimap A = C^-[\top]$ for some type context C^- where the hole marks out the (negative) occurrence of the $\top$ in question. We depict the two links that distinguish the two strategies as follows:

$$\top \cdots\cdots\cdots\cdots a \cdots\cdots b$$

$$\top \cdots\cdots\cdots\cdots a \cdots\cdots b$$

We call the two strategies σ_a and σ_b (for the game $C^-[G_*]$). Consider the new linkage which is obtained by augmenting the links that are common to both σ_a and σ_b by the following:

$$\top_a \otimes \top_b \cdots\cdots\cdots\cdots a \cdots\cdots b$$

Call the strategy defined by the augmented linkage σ (for the game $C^-[G_{*_a} \otimes G_{*_b}]$). We need the following lemma:

Lemma 46. *The strategy σ is conditionally exhausting.*

Proof. Equivalently, we prove that the new linkage defines a correct essential net. Firstly, let us note that in the new essential net there is no path going through both of the distinguished unit links. Suppose the contrary. Then there must be a path from either $\top_a$ to b or from $\top_b$ to a. From the structure of the essential net, it then follows that there are paths either from $\top_b$ to b or from $\top_a$ to a. Therefore there are cycles in the nets corresponding to σ_a or σ_b—a contradiction as both of them define correct essential nets.

By the previous remark, paths in the new net are morally paths from the old nets (without any combinations). Hence acyclicity and Condition L hold. □

Let $\vdash u : C^-[\top \otimes \top]$ be the $\mathbf{IMLL}^\flat$ term that corresponds to σ by Theorem 40. To complete the proof, consider the two conditionally exhausting strategies for the game

$$C^-[G_{*_a} \otimes G_{*_b}] \multimap C^-[G_*],$$

which behave everywhere in the same (copycat) manner except that they differ in which one of the two $*$'s on the left triggers the $*$ on the right. Call the $\mathbf{IMLL}^\flat$ terms corresponding to these strategies (via Theorem 40)

$$y : C^-[\top \otimes \top] \vdash l : C^-[\top] \quad \text{and} \quad y : C^-[\top \otimes \top] \vdash r : C^-[\top],$$

respectively. By an easy structural induction, we have

$$y : C^-[\top \otimes \top] \vdash l = r : C^-[\top]$$

is provable in **IMLL**. Thus $\vdash l\{u/y\} = r\{u/y\} : C^-[\top]$ is also provable in **IMLL**. But since both $\boldsymbol{\lambda} x.s = l\{u/y\}$ and $r\{u/y\} = \boldsymbol{\lambda} x.t$ are provable in $\mathbf{IMLL}^\flat$ (as $\mathscr{G}_{ce}$ is a fully complete model of $\mathbf{IMLL}^\flat$), we have $\Gamma \vdash s = t : A$ is **IMLL**-provable as required. □

Consequently we have the main result of the paper.

Theorem 47 (Full completeness). (i) *$\mathcal{G}_a$ is a fully complete model of* **IMLL**.

(ii) *$\mathcal{G}_a$ is isomorphic to the autonomous category freely generated from the discrete graph whose vertices are $a, b, c, \ldots$.*

10. Coherence and the tower of units problem

By the Categorical Type Theory Correspondence and by the Full Completeness Theorem 47, we can use our games to examine coherence properties of free autonomous categories. As an immediate corollary of the Full Completeness Theorem, we have the following.

Corollary 48 (Coherence 2). *If a sequent of* **IMLL** *is linearly balanced and contains one negative occurrence of the unit, then it has at most one $=$-equivalence class of proofs.* □

Conjecture 49 (Coherence 3). *If Γ is linearly balanced and contains only two negative occurrences of the tensor unit, then there is at most one $=$-equivalence class of proofs of Γ.* □

10.1. Triple unit problem

There are sequents for which we cannot hope to prove "coherence". The standard example is the Triple Unit Problem (see e.g. [12]), but here we recast it using the language of **IMLL**, via the Categorical Type Theory Correspondence. If a is an atomic type not equal to $\top$, it is known that there are exactly two inequivalent ways of proving

$$((a \multimap \top) \multimap \top) \multimap \top \vdash ((a \multimap \top) \multimap \top) \multimap \top$$

in **IMLL**. See [14] for a proof using the syntax of **IMLL**. This fact does not depend on the number of occurrences of unit of positive polarity: as Francois Lamarche has observed, there are exactly two inequivalent ways of proving

$$((a \multimap \top) \multimap (x \multimap x)) \multimap \top) \vdash ((a \multimap (y \multimap y)) \multimap \top) \multimap (z \multimap z),$$

where x, y and z are atomic.

10.2. The tower of units problem

Take any type A. We define inductively:

$$A(0) = A,$$
$$A(n+1) = A(n) \multimap \top.$$

We aim to compute the number of maps of the type $a(n) \to a(n)$, where $A = a$ is atomic, in a free autonomous category. There are two cases.

Suppose n is odd. Enumerate all the unit occurrences starting from the right. The two occurrences of the atomic type will be referred to as a_L and a_R. By P-exhaustion a_R must be reached at some point. There is no choice as to the move that triggers a_R—it must be a_L. But for that to happen, all other moves must be played first—this is a consequence of the rules of the game. Therefore the position ending in a_R (coupled with the fact that strategies are deterministic and contain just one maximal position) tells us everything about the strategy. Besides, each such must be built using two-move blocks

$$*_2*_3, \quad *_4*_5, \quad \cdots \quad *_{2n-2}\,*_{2n-1};$$

$*_{2n}$ always precedes a_L and every play is started by $*_1$:

$$*_1 \underbrace{\cdots\cdots\cdots\cdots\cdots}_{2n-2} *_{2n} a_L a_R.$$

The blocks from one side must appear in the order dictated by their subscripts, the way the two sides interleave is arbitrary, though. There are $n-1$ places for the blocks to fill in and it suffices to choose which places will be occupied by each side. There are $(n-1)/2$ blocks on each side, so the number of conditionally exhausting strategies is $\binom{n-1}{\frac{n-1}{2}}$. None of them can be equated in $\mathcal{G}_a$ as they differ by at least two links.

The case where n is even is similar. Each position begins in $*_1$ as before but ends in $*_n a_R a_L$. The building blocks are:

$$*_2*_3, \quad \ldots, \quad *_{n-2}*_{n-1}, \quad *_{n+1}*_{n+2}, \quad \ldots, \quad *_{2n-1}\,*_{2n}\,.$$

There are $(n-2)/2$ and $n/2$ of them on the left and right, respectively. The total number of strategies is thus $\binom{n-1}{\frac{n}{2}}$. We summarize our findings as follows.

Theorem 50. *Let* $\Theta = A(n) \multimap A(n)$.

(i) *If* $A \neq \top$, *there are at least* $\binom{n-1}{\frac{n}{2}}$ *inequivalent proofs of* Θ *if* n *is even and* $\binom{n-1}{\frac{n-1}{2}}$ *if* n *is odd.*

(ii) *The estimate becomes accurate if* A *is linearly balanced and* (*is isomorphic to an object that*) *does not contain any unit.* □

We illustrate the theorem by showing that the two strategies in the Triple Unit Problem coincide in our model if and only if $A = \top$. In fact, in that case, we have as many as eight conditionally exhausting strategies, as shown in Fig. 8; note that the last one in the figure is the canonical identity. The way the eight strategies are related by $\sim$ is depicted in Fig. 7: the numbers therein refer to the order in which the strategies appear in Fig. 8.

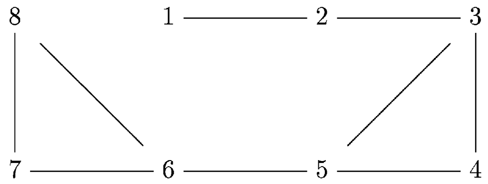

Fig. 7. The $\sim$-relation between the eight strategies in Figure 8.

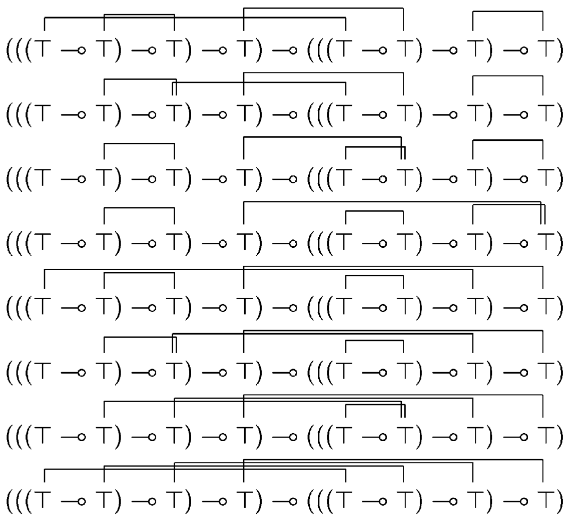

Fig. 8. Eight conditionally exhausting strategies.

Acknowledgements

We are grateful to Hanno Nickau and the anonymous referees for helpful comments on a draft of the paper. The first author is supported by a University of Oxford Scatcherd European Scholarship and a British Overseas Research Students Award 98032135. This work is partially supported by EU TMR AppSem and LINEAR.

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