Rooted pointwise limit of graphs

Paper [114] gives an asymptotic characterization of connected finite vertex-transitive graphs of bounded (orientable or non-orientable) genus. All but a finite number of such graphs are embeddable on the torus or the Klein bottle.

The proof goes via an excursion into the infinite. We consider the rooted pointwise limit of a class of such graphs. This limit is planar. We classify the limit as having one, two, or infinitely many ends. We show that the cases with one end give an archimedean tiling of the plane and therefore correspond to toridal tessellations; the cases of two ends are either toroidal or correspond to a tessellation of the the Klein bottle; and the cases with infinitely many ends do not occur: if the limit of a sequence of finite connected vertex-transitive graphs has infinitely many end then the finite graphs in question have unbounded Hadwiger numbers by a sphere packing argument. (The Hadwiger number is the size of the largest clique minor of the graph.)

This seems to be one of the earliest papers to use the interplay between finite and infinite graphs via the rooted pointwise limit.

A subsequent, yet unpublished result, mentioned in my chapter of the *Handbook of Combinatorics* cited below, gives an asymptotic characterization of the connected vertex-transitive graphs of bounded Hadwiger number. These graphs are either ring-like or toroidal. A graph is ring-like if its vertices can be arranged in a circular order such that every edge skips a bounded number of vertices along the circle.

The proof proceeds along similar lines as the bounded-genus case. Infinitely many ends cannot occur for the reason mentioned. Two ends correspond to a sequence of ring-like graphs. In the case of one end, it can be shown that the limit graph is an archimedean tessellation of the euclidean or the hyperbolic plane. The case of the hyperbolic plane can be shown to lead to unbounded Hadwiger number of the finite graphs in question, using another sphere packing argument that takes advantage of the fact that the circumference of a disc grows exponentially as a function of the radius. What is left is the case of the euclidean tessellation, leading to toroidal graphs.

[114] László Babai. Vertex-transitive graphs and vertex-transitive maps. *J. Graph Theory*, 15(6):587–627, 1991. doi:10.1002/jgt.3190150605.

[145] László Babai. Automorphism groups, isomorphism, reconstruction. In R. L. Graham, M. Grötschel, and L. Lovász, editors, *Handbook of Combinatorics*, chapter 27, pages 1447–1540. North-Holland – Elsevier, 1995.