Constraints on low order models: The cost of simplicity

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Abstract

Low degree of freedom models have been proposed to understand the transition to turbulence and the coherent structures which form. The low order model on which the Self Sustaining Process [10] for turbulence in Couette flow is examined with respect to mean conservation laws of Fourier truncations Aspects of the non-dissipative limits of these models are studied. Transport equations and conservation laws are derived for Energy, vorticity and helicity. Connections are made between low order models and conservation laws and Lie algebras

1 Introduction

The transition from laminar to turbulent flow is of interest in bounded shear flow geometries. Laminar flows are advection free and mix only by molecular diffusion while turbulent flows are characterized by drastic increases in mixing rates. The change in mixing regimes is due to advective transport accessible to the turbulent state. Some flows can transition through a linear instability of the laminar flow. In contrast systems like plane Couette flow which are stable to infinitesimal perturbations exhibit transitions to turbulence for finite amplitude perturbations.

1.1 Equations of Motion

The governing equations of fluid flow can be constructed through two very different approaches. The most fundamental method of generating equations of fluid motion begins with a continuously labelled set of fluid particles from which a Hamiltonian is constructed[1]. From this Hamiltonian formulation conservation laws may in principle be derived using Noether's theorem. For most fluids it is difficult to write an expression for the Hamiltonian in this manner as the natural co-ordinate system is advected with the particles.

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[1]For a discussion of Hamiltonian dynamics of fluids see [4, 8]

An alternate approach constructs a set of partial differential equations which include specific conservation laws. This approach is also more amenable to inclusion of empirical equations of state and stress strain relations. The Navier-Stokes equations

$$\frac{\partial v}{\partial t} + v \cdot \nabla v = -\frac{1}{\rho}\nabla p + \frac{1}{\mathrm{R}}\nabla^2 v_i \tag{1}$$

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \rho v = 0 \tag{2}$$

are derived in this manner, using assumptions about diffusion of fluid properties and conservation of mass and momentum. The parameter R is the Reynolds number which gives a measure of the relative influence of diffusion and advection. The ratio of advective and diffusive transports is given by the Reynolds number[7, 9, 6]

$$\mathrm{R} = \frac{UL}{\nu} \tag{3}$$

where U and L are characteristic velocity and length scales and ν is the molecular diffusivity.

These two modes of derivation can be bridged starting with governing equations from application of conservation laws then seeking to find a Hamiltonian system which is consistent with those dynamics. This requires an expression for the Hamiltonian H and a Poisson bracket operator $\{A, B\}$. The Poisson bracket is implicitly defined by the time evolution operator

$$\frac{\partial}{\partial t} A = \{A, H\} \tag{4}$$

For example in an ideal fluid is governed by the Euler equations

$$\begin{aligned}
\frac{\partial v}{\partial t} + v \cdot \nabla v &= -\frac{1}{\rho}\nabla p && (5)\\
\frac{\partial \rho}{\partial t} + \nabla \cdot \rho v &= 0 && (6)\\
\frac{\partial s}{\partial t} + v \cdot \nabla s &= 0 && (7)
\end{aligned}$$

and have a Hamiltonian takes of the form

$$H[v, \rho, s] = \iiint \frac{1}{2}\rho v \cdot v + \rho U(\rho, s)\ \mathrm{d}^3 r \tag{8}$$

where v is the velocity, ρ is density, s is entropy density and U is internal energy. The Poisson bracket for this system is found explicitly by [5, 4]

$$\begin{aligned}
\{A, B\} = &-\iiint \frac{\delta A}{\delta \rho}\nabla \cdot \frac{\delta B}{\delta v} - \frac{\delta B}{\delta \rho}\nabla \cdot \frac{\delta A}{\delta v} \\
&+ \frac{\nabla \times v}{\rho} \cdot \frac{\delta B}{\delta v} \times \frac{\delta A}{\delta v} \\
&+ \frac{\nabla s}{\rho} \cdot \left(\frac{\delta A}{\delta s}\frac{\delta B}{\delta v} - \frac{\delta B}{\delta s}\frac{\delta A}{\delta v} \right) \mathrm{d}^3 r
\end{aligned} \tag{9}$$

where $\frac{\delta A}{\delta v}$ is the functional derivative. It is from this middle road approach we take the inspiration for analysis of low order models of turbulence.

1.2 Flow parameters

Since the ideal fluid (5) is the limit as $\mathrm{R} \to \infty$ of the viscous fluid (1) it is instructive to describe some of the characteristics such flows. Before continuing the analysis we further restrict ourselves to incompressible flows. This replaces the density evolution equation with the requirement that velocity be divergence free $\nabla \cdot v = 0$. To highlight the differences between incompressible solutions to (5) and (1) we will focus on the evolution of three quantities. Kinetic energy $\frac{1}{2}v^2$, vorticity $\omega = \frac{1}{2}\nabla \times v$, and helicity $\omega \cdot v$ are governed by the transport equations

$$\begin{aligned} \frac{\partial}{\partial t}\frac{v^2}{2} &= \nabla_i \left(-\frac{v_i P}{\rho} + \frac{1}{\mathrm{R}}\nabla_i \frac{v^2}{2} \right) - \frac{(\nabla_i u_j)^2}{\mathrm{R}} && (10) \\ \frac{\partial}{\partial t}\omega_i &= \nabla_j \left(\omega_j v_i - v_j \omega_i + \frac{1}{\mathrm{R}}\nabla_j \omega_i \right) && (11) \\ \frac{\partial}{\partial t} v_i \omega_i &= \nabla_i \left(\frac{\rho v^2 - 2P}{2\rho}\omega_i - v_i v_j \omega_j + \frac{\nabla_i \omega_j v_j}{\mathrm{R}} \right) - \frac{2\left(\nabla_j \omega_i\right)\left(\nabla_j v_i\right)}{\mathrm{R}}. && (12) \end{aligned}$$

For the inviscid case the transport equations are a flux divergence and these quantities are conserved. When viscosity is introduced energy is damped and vorticity is made more homogeneous. The influence of diffusion on helicity is more complex, in addition to a homogenizing diffusive term there is an additional term whose sign is indeterminate.

At large but finite Reynolds number diffusion is weak and only has influence at very small length scales. This separation of scales is the basis of the Kolmogorov hypothesis where at intermediate scales the motion self-organizes so that there is a constant energy dissipation rate ϵ. The Kolmogorov length scale

$$\eta = \sqrt[4]{\nu^3/\epsilon} \propto \sqrt[4]{\frac{1}{\mathrm{R}}} \tag{13}$$

is the scale at which diffusion can dissipate all the energy input and sets a lower bound on the size of flow persistent features[9, 6].

1.3 Truncation

For very viscous flows the range of scales of motion available are limited, this has led to the exploration of low dimensional models of turbulent transitions [10, 3, 2]. Using a spectral decomposition of the flow these low order models retaining a subset of the modes which represent motion with the largest length scales and whose dynamics will be least damped. If density ρ is assumed to be constant (1) can be written in spectral form

$$\frac{\partial v(k)}{\partial t} + v(k) * ikv(k) = -\frac{1}{\rho}ikp(k) - \frac{1}{\mathrm{R}}k^2 v(k) \tag{14}$$

where $*$ is the convolution operator. Truncation changes the behaviour of the convolution operator by restricting the domain available in the spectral space to a set of amplitudes Z_i. The result is a finite set of non-linear differential equations for the amplitude of the chosen modes. It is important to emphasize that the choice of which modes enter into the truncation can greatly effect the behaviour of the low dimensional system.

Projecting (1) into Fourier space $\hat{u}_i(k)$ imposes a strong constraint on the motion through phase space

$$\frac{\partial}{\partial \hat{u}_j(k)} \frac{\mathrm{d}\hat{u}_i(k)}{\mathrm{d}t} = 2\mathrm{i}\left(\delta_{ij} - \frac{k_i k_j}{k^2}\right)\hat{u}_m(0)k_m - \frac{1}{\mathrm{R}}k^2\delta_{ij}. \tag{15}$$

Trajectories in the inviscid limit $\frac{1}{\mathrm{R}} \to 0$, with no mean flow $\hat{u}(0) = 0$ are divergence free. These divergence free trajectories additionally satisfy a Detailed Liouville where each term in the sum (15) is separately zero. This Detailed Liouville behaviour in Fourier space has been noted by [1] without the constraint on mean flow.

Some additional properties of the trajectories are low dimensional analogues of Flux-divergence conservation laws of the full partial differential equation. If Fourier modes are used for the decomposition there are some basic properties which all truncations will share. The volume average of these transport equations can be used to form additional constraints on the amplitudes of a Fourier decomposition. By Parceval's theorem mean kinetic energy is simply the sum of squares of Fourier amplitudes and must evolve according to

$$\sum_{i=1}^{N} Z_i \dot{Z}_i = -\frac{1}{\mathrm{R}}\sum_{i=1}^{N} \kappa_i^2 Z_i^2 \tag{16}$$

where κ_i is the wave vector for mode i. Similarly mean vorticity $\sum_{i=1}^{N} Z_i\omega(Z_i)$ and mean helicity $\sum_{i=1}^{N} Z_i\varpi(Z_i)$ must evolve according to

$$\sum_{i=1}^{N} \dot{Z}_i\omega(Z_i) = -\frac{1}{\mathrm{R}}\sum_{i=1}^{N} \kappa_i^2 Z_i\omega(Z_i) \tag{17}$$

$$\sum_{i=1}^{N} \dot{Z}_i\varpi(Z_i) = -\frac{2}{\mathrm{R}}\kappa_i\omega_j(Z) * \kappa_i u_j(Z) \tag{18}$$

where $u(Z_i)$, $\omega(Z_i)$, and $\varpi(Z_i)$ are the velocity, vorticity and helicity in mode Z_i when it has unit amplitude. Mean vorticity dynamics rarely contribute to low order truncations, it is common to have no modes with mean vorticity [2], one mode out of eight [10], or two modes out of nine [3]. Helicity dynamics are absent in low order models where all the included modes have zero helicity [10, 3, 2]. The absence of helicity is due to the choice of modes not the boundary conditions, in the domain used by [10, 3] a helical mode with similar wave-numbers to those included in the truncation is of the form

$$\vec{u} = \begin{pmatrix} \cos^2\pi y \sin\pi y \\ 0 \\ \cos\pi y \sin^2\pi y \end{pmatrix} \tag{19}$$

where the velocity spirals between the boundaries.

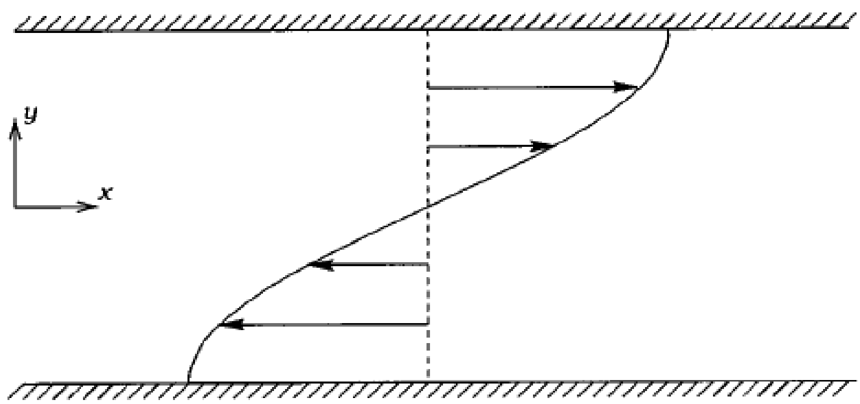

Figure 1: Mean flow mode M from [10]

2 Examination of Low Order Models

Analogously to how [5, 4] take the continuum equations for momentum balance in an ideal fluid we hope to connect the low order models of [10] to an underlying Hamiltonian system in the inviscid limit. With that goal in mind the necessary conditions of truncations are checked for the four and five dimensional systems proposed and both systems are found to behave in a manner inconsistent with a truncation. In the inviscid limit the fourth order system has atracting regions in phase space and can not be a Hamiltonian system.

2.1 Detailed Liouville

In the inviscid limit any Fourier truncation is expected to be divergence free in phase space. This Liouvillian character of trajectories implies there can not be any purely attracting regions in phase space once the effects of viscosity have been removed. Because the parent model is a truncation of Fourier modes and there is no mean flow there is the stronger condition that the trajectories have a Detailed Liouville behaviour (15). Using Z_i for the amplitude of the i^{th} Fourier mode in the truncation

$$\frac{\partial \dot{Z}_i}{\partial Z_i} = 0 \text{ no sum on } i. \tag{20}$$

The simplest low order model we consider is the fourthorder model from [10]. This model has a mean flow M figure 1 and three additional modes charachterized as streaks U, Stream-wise vorticies, and a Streak Instability W which are combined into a Self Sustaining Process figure 2. The governing equations

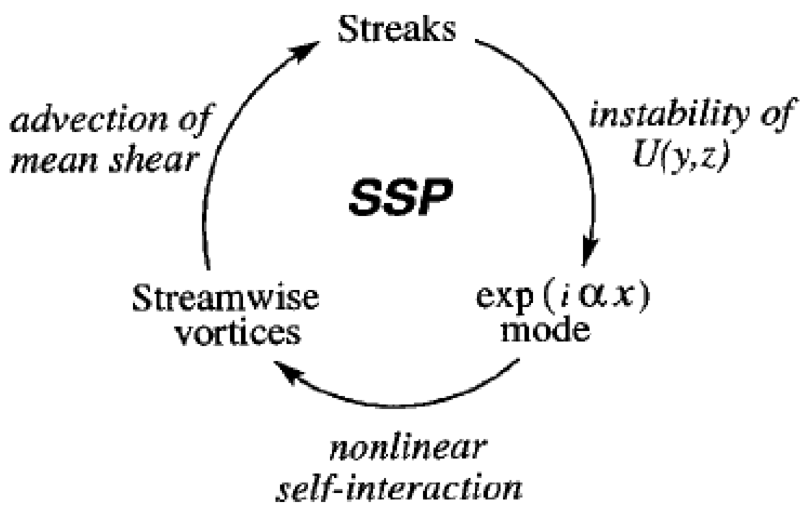

Figure 2: Self sustaining process from [10]

$$\begin{aligned}
\text{Mean shear } \dot{M} &= -UV + \Sigma_m W^2 - \frac{1}{R}(M-1) && (21)\\
\text{Streaks } \dot{U} &= +MV + \Sigma_u W^2 - \frac{\kappa_u^2}{R} U && (22)\\
\text{Stream-wise Rolls } \dot{V} &= +\Sigma_v W^2 - \frac{\kappa_v^2}{R} V && (23)\\
\text{Streak Instability } \dot{W} &= -W\left(\Sigma_m M + \Sigma_u U + \Sigma_v V\right) - \frac{\kappa_w^2}{R} W && (24)
\end{aligned}$$

include a source term in the mean flow $\frac{1}{\mathrm{R}}(M-1)$ to model the input of energy from outside the system. The coupling constants $\Sigma_{m,u,v}$ arise from the convolution of the truncated modes. and are functions of the particular wave-numbers $\kappa_{u,v,w}$ chosen in the truncation. The equations have been made non-dimensional to minimize the number of constants. The fourth order model of [10] has an attracting orbit in the $\frac{1}{\mathrm{R}} \to 0$ limit is shown in figure 3

$$(M, U, V, W) = (M_0 \cos V_0 t, M_0 \sin V_0 t, V_0, 0). \quad (25)$$

This convergent behaviour in phase space indicated that the simplifying assumptions used to reduce the model from an eighth order truncation introduced non-physical dissipation.

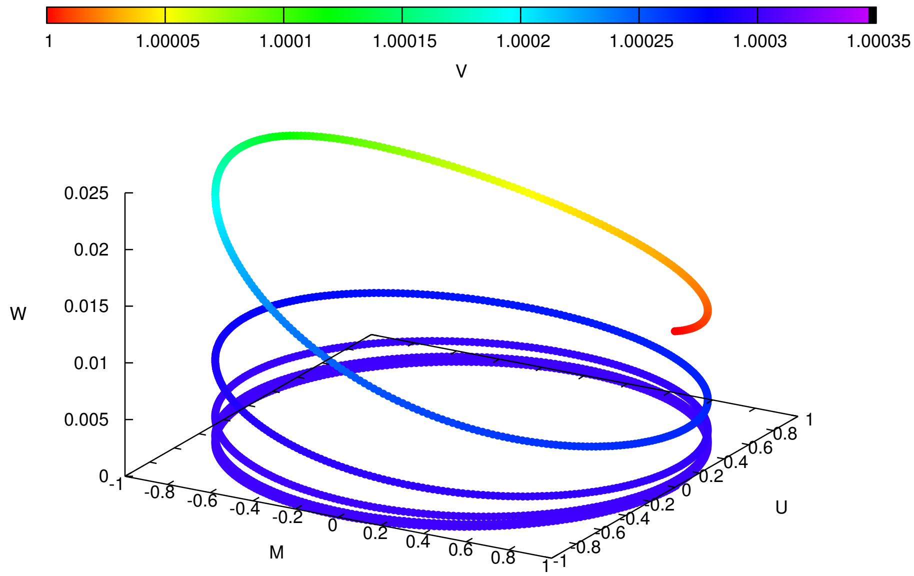

Figure 3: Attracting orbit of [10] 4 dimensional model in the limit $\frac{1}{\mathrm{R}} \to 0$. M, U, W are shown in projection and V is colored

The fifth order parent model

$$\text{Mean shear } \dot{M} = -UV + \Sigma_m AE - \frac{1}{R}(M-1) \tag{26}$$

$$\text{Streaks } \dot{U} = +MV + \Sigma_{ua}A^2 + \Sigma_{ue}E^2 - \frac{\kappa_u^2}{R}U \tag{27}$$

$$\text{Rolls } \dot{V} = +\Sigma_v AE - \frac{\kappa_v^2}{R}V \tag{28}$$

$$\dot{A} = -\frac{E}{2}(\Sigma_m M + \Sigma_v V) - \Sigma_{ua}UA - \frac{\kappa_a^2}{R}A \tag{29}$$

$$\dot{E} = -\frac{A}{2}(\Sigma_m M + \Sigma_v V) - \Sigma_{ue}UE - \frac{\kappa_e^2}{R}E \tag{30}$$

is non-divergent but fails the more stringent requrement of being Detailed Liouvillian (20). Using the fifth order model from [10] as a starting point a similar set of equations which satisfies Detailed Liouville (20) is constructed

$$\begin{aligned}
\text{Mean shear } \dot{M} &= -UV + \Sigma_m AE - \frac{1}{R}(M-1) \\
\text{Streaks } \dot{U} &= +MV + \Sigma_u AE - \frac{\kappa_u^2}{R}U \qquad (31)\\
\text{Rolls } \dot{V} &= +\Sigma_v AE - \frac{\kappa_v^2}{R}V \\
\dot{A} &= -\frac{E}{2}(\Sigma_m M + \Sigma_u U + \Sigma_v V) - \frac{\kappa_a^2}{R}A \qquad (32)\\
\dot{E} &= -\frac{A}{2}(\Sigma_m M + \Sigma_u U + \Sigma_v V) - \frac{\kappa_e^2}{R}E \qquad (33)
\end{aligned}$$

where A^2 and E^2 have been replaced by a term proportional to AE in (27) and terms proportional to UA in (29) have been replaced by UE and vice-versa in (30). In the inviscid limit this new set of equations satisfies Detailed Liouville (20), as well as conservation of energy (16), and vorticity (17).

The stability of the laminar state equilibrium, where $M = 1$ and all others are zero, can be explored using energy stability. The laminar state L_i is globally stable with respect to the energy while

$$\frac{\mathrm{d}}{\mathrm{d}t}(Z_i - L_i)^2 \leq 0 \qquad (34)$$

for any system state Z_i. This criterion is dependent on the value of R, in fact the laminar state is stable only if both

$$\begin{aligned}
\mathrm{R} &\leq 2\left|\kappa_u \kappa_v\right| \qquad (35)\\
\mathrm{R} &\leq 2\left|\frac{\kappa_a \kappa_e}{\Sigma_m}\right| \qquad (36)
\end{aligned}$$

hold. The other equilibria of the system of equations can be found by solving the roots of a 5^{th} order polynomial, but the solution is not amenable to testing limiting behaviour. The equilibria criteria can be simplified into the relations

$$\begin{aligned}
U &= -\frac{\Sigma_m \Sigma_v M^2 + \frac{\Sigma_m \Sigma_u \kappa_v^2 - 2\kappa_a \kappa_e \Sigma_v}{R} M - \frac{2\kappa_a \kappa_e \Sigma_u \kappa_v^2}{R^2}}{\Sigma_u \Sigma_v M + \frac{\kappa_u^2 \Sigma_v^2 + \Sigma_u^2 \kappa_v^2}{R}} \qquad (37)\\
V &= \frac{\kappa_u^2 \Sigma_v U}{\Sigma_v R M + \Sigma_u \kappa_v^2} \qquad (38)\\
E^2 &= -\frac{\kappa_a \kappa_v^2 V}{\kappa_e \Sigma_v R} \qquad (39)\\
A^2 &= \frac{\kappa_e^2}{\kappa_a^2} E^2 \qquad (40)
\end{aligned}$$

in addition to the trivial solution $M = U = V = A = E = 0$. From these relations and (26)

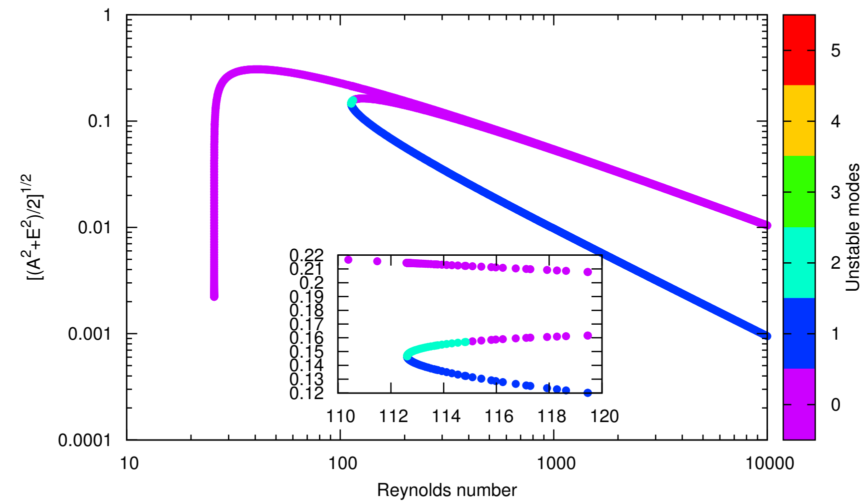

Figure 4: Equilibrium solutions as a function of Reynolds number, colored by number of unstable modes

the limiting behaviour as R $\rightarrow \infty$

$$
\begin{aligned}
M &\rightarrow \frac{1}{1+\kappa_u^2 \frac{\Sigma_m^2}{\Sigma_u^2}} && (41)\\
U &\rightarrow -\frac{\Sigma_m}{\Sigma_u} M && (42)\\
V &\rightarrow -\frac{\Sigma_m \kappa_u^2}{\Sigma_u R} && (43)\\
E^2 &\rightarrow \frac{\kappa_a \Sigma_m \kappa_v^2 \kappa_u^2}{\kappa_e \Sigma_u \Sigma_v R^2} && (44)\\
A^2 &\rightarrow \frac{\kappa_e \Sigma_m \kappa_v^2 \kappa_u^2}{\kappa_a \Sigma_u \Sigma_v R^2} && (45)
\end{aligned}
$$

which differs from both the zero and laminar solutions. At finite Reynolds number these solutions come in families where the sign of E, and A are arbitrary when (39) has real roots.

To continue to characterize the behavior of the system it is helpful to use numerical methods and integrate the equations of motion. Since the equilibria are insensitive to the sign of A or E and both the trivial and laminar states have $A = E = 0$ figure 4 plots $\sqrt{\frac{1}{2}\left(A^2 + E^2\right)}$ of the equilibria against Reynolds number. The laminar equilibrium becomes an unstable equilibrium at a Reynolds number between 20 and 30. The break in the graph in that region is due to the discrete step size taken in Reynolds number. Two additional unstable equilibria arise near R $\approx$ 113 unlike the equilibrium near R $\approx$ 20 which is stable.

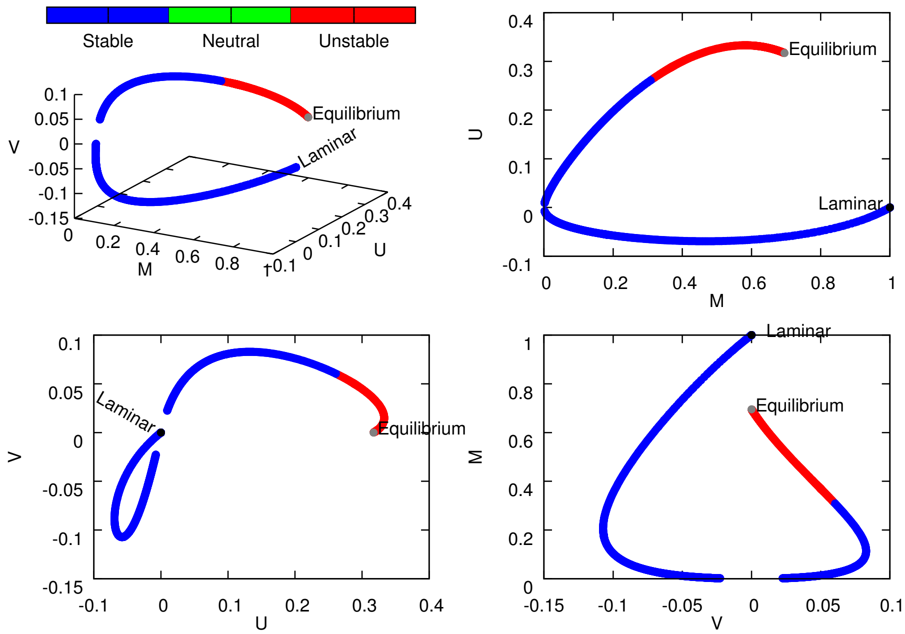

Figure 5: Equilibrium solutions colored by stability

One equilibrium remains unstable for all Reynolds numbers while the other becomes stable for increasing Reynolds number.

The nature of this transition can be more easily seen in figures 5 and 6 where the equilibria are plotted in M, U, V space. The two curves show the two disjoint families of equilibrium of solutions. As Reynolds number is increased the laminar state smoothly transitions to a state of no motion. In contrast at moderate Reynolds number another set of solutions come into existence, an unstable equilibrium smoothly approaches the inviscid equilibrium state as Reynolds number increases. The other equilibrium is initially very unstable ubt quickly become stable and smoothly approaches the state of no motion. This show two co-existing stable equilibria which approach each other in the limit of large Reynolds number.

2.2 Hamiltonian

Having modified the five dimensional system to satisfy Detailed Liouville we now seek to find a Hamiltonian system from which it can be derived. The construction is simplified by the quadratic form of the energy

$$H = M^2 + U^2 + V^2 + A^2 + E^2 \tag{46}$$

where the H is chosen to emphasize that in the inviscid system the Hamiltonian is the energy. Now we need a co-ordinate transformation so that we can recover the equations of motion using a Poisson bracket

$$\dot{Z}^i = \left\{ Z^i, H \right\}. \tag{47}$$

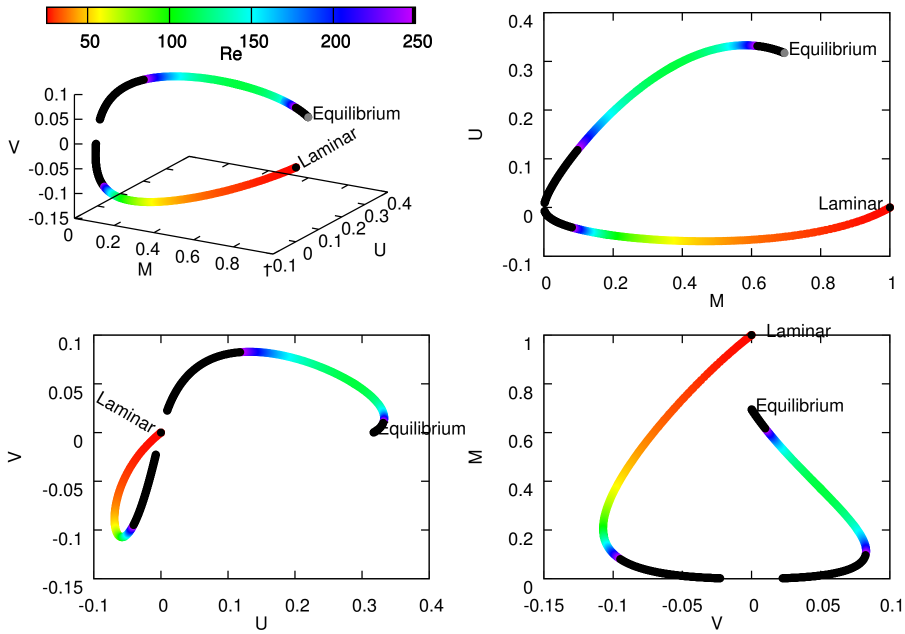

Figure 6: Equilibrium solutions colored by Reynolds number

For finite dimension Hamiltonian systems the Poisson bracket used for time evolution can be written in a general from

$$\{f, g\} = \frac{\partial f}{\partial Z^i} J^{ij} \frac{\partial g}{\partial Z^j} \tag{48}$$

where J^{ij} is an anti-symmetric tensor which satisfies the Jacobi identity

$$S^{ijk} = J^{im} \frac{\partial J^{jk}}{\partial Z^m} + J^{jm} \frac{\partial J^{ki}}{\partial Z^m} + J^{km} \frac{\partial J^{ij}}{\partial Z^m} + = 0 \tag{49}$$

Because we are using a quadratic Hamiltonian and are enforcing Detailed Liouville the tensor J^{ij} is of the form

$$J^{ij} = C^{ij}_k Z^k \tag{50}$$

where C^{ij}_k form a Lie algebra. For four dimensional systems there are only two Lie algebras which give rise to full rank tensors J^{ij}, the two archetypal forms of the J tensor are

$$J^{ij} = \begin{pmatrix} 0 & -Z_1 & 0 & 0 \\ Z_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -Z_3 \\ 0 & 0 & Z_3 & 0 \end{pmatrix} \text{or} \begin{pmatrix} 0 & -Z_3 & 0 & 0 \\ Z_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & -Z_1 \\ 0 & 0 & Z_1 & 0 \end{pmatrix} \tag{51}$$

but neither admit linear co-ordinate transformations which reproduce the characteristics of the fourth order model of [10] such as non-normality. For the five dimensional case the possible Lie algebras have not been exhausted but no J^{ij} which satisfies the Poisson bracket has been found which satisfies the Jacobi identity (49).

3 Conclusion

A non-dimensional version of the low order models in [10] have been shown to introduce an additional damping not present in the continuous system. Hamiltonian generalization of the fourth order model has been been excluded using properties of Lie algebras. The fifth order model [10] has been shown to violate the Detailed Liouville criterion, a simple substitution was introduced into the fifth order model to recover Detailed Liouville. The behaviour of this modified fifth order model is explored in detail and the stability and multiplicity of the equilibria is shown for a range of Reynolds numbers. No conclusive statement is made about the existence of a Hamiltonian parent model for the models of order higher than four. A trial form for the anti-symmetric tensors J^{ij} was found for both fifth order systems was found which satisfied time evolution as a Poisson bracket. The candidate tensors fail the Jacobi identiy, the time evolution only uniquely determines J^{ij} for systems of order three and less and no general statement can be made from the failure of of the Jacobi identity for these particular tensors.

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