Overview of turbulent shear flows

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A brief review of basic concepts and folklore in shear turbulence.

1 Reynolds decomposition

Experimental measurements show that turbulent flows can be decomposed into well-defined averages plus fluctuations, $\mathbf{v} = \bar{\mathbf{v}} + \mathbf{v}'$. This is nicely illustrated in the *Turbulence* film available online at `http://web.mit.edu/hml/ncfmf.html` that was already mentioned in Lecture 1.

In experiments, the average $\bar{\mathbf{v}}$ is often taken to be a time average since it is easier to measure the velocity at the same point for long times, while in theory it is usually an *ensemble average* — an average over many realizations of the same flow. For numerical simulations and mathematical analysis in simple geometries such as channels and pipes, the average is most easily defined as an average over the homogeneous or periodic directions.

Given a velocity field $\mathbf{v}(x, y, z, t)$ we define a mean velocity for a shear flow with laminar flow $U(y)\hat{\mathbf{x}}$ by averaging over the directions (x, z) perpendicular to the shear direction y,

$$\bar{\mathbf{v}} = \lim_{L_x, L_z \to \infty} \frac{1}{L_x L_z} \int_{-L_z/2}^{L_z/2} \int_{-L_x/2}^{L_x/2} \mathbf{v}(x, y, z, t) \mathrm{d}x \mathrm{d}z = \overline{U}(y, t)\hat{\mathbf{x}}. \tag{1}$$

The mean velocity is in the x direction because of incompressibility and the boundary conditions. In a pipe, the average would be over the streamwise and azimuthal (spanwise) direction and the mean would depend only on the radial distance to the pipe axis, and (possibly) time. Now we apply this averaging to the incompressible Navier-Stokes equations,

$$\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} = -\nabla p + \nu \nabla^2 \mathbf{v} \quad , \quad \nabla \cdot \mathbf{v} = 0. \tag{2}$$

Using the incompressibility condition $\nabla \cdot \mathbf{v} = 0$ we can rewrite the momentum equation in conservative form,

$$\frac{\partial \mathbf{v}}{\partial t} + \nabla \cdot (\mathbf{v}\mathbf{v}) = -\nabla p + \nu \nabla^2 \mathbf{v}. \tag{3}$$

We split the flow into a mean part and a fluctuation,

$$\mathbf{v} = \overline{U}(y, t)\hat{\mathbf{x}} + \mathbf{v}' \tag{4}$$

Averaging the Navier-Stokes equations (3) over x and z yields the *mean flow equation*

$$\frac{\partial \overline{U}}{\partial t} + \frac{\partial \overline{uv}}{\partial y} = -\frac{dP_0}{dx} + \nu \frac{\partial^2 \overline{U}}{\partial y^2}, \tag{5}$$

where $\mathbf{v}' = (u, v, w)$ in cartesian coordinates and the overline $\overline{()}$ denotes the average over x and z. To arrive at this mean flow equation, we used the divergence theorem and the fact that $\overline{\mathbf{v}'} = 0$. The mean pressure gradient $\overline{\partial P/\partial x} = dP_0/dx$ is not zero in pressure-driven channel and pipe flows where the pressure has the form $P = P_0(x) + p(\mathbf{r}, t)$ with dP_0/dx a fixed constant and $p(\mathbf{r}, t)$ is the flow induced pressure required to maintain incompressibility. For fixed flux, as in the Mullin pipe flow experiments [11], the mean pressure gradient can fluctuate in time to maintain the total mass flux.

For sufficiently large domains and sufficiently large Reynolds number, the flow in a pipe or channel is *statistically steady*, so the mean values are independent of t.[1] For statistically steady flow, the mean flow equation (5) reduces to

$$\frac{d\overline{uv}}{dy} = -\frac{dP_0}{dx} + \nu \frac{d^2 \overline{U}}{dy^2}. \tag{6}$$

Integrating (6) from the bottom wall at $y = -h$ to the top wall at $y = h$ gives $0 = -2h\frac{dP_0}{dx} + \nu \left.\frac{d\overline{U}}{dy}\right|_h - \nu \left.\frac{d\overline{U}}{dy}\right|_{-h}$ since $u = v = 0$ at the walls, leading to

$$-\frac{dP_0}{dx} = \frac{\tau_w}{h} \tag{7}$$

where $\tau_w = \nu \, d\overline{U}/dy\big|_{-h} = -\nu \, d\overline{U}/dy\big|_h$ is the shear stress on the bottom wall and top wall, by symmetry.

Plane Couette flow is driven by moving walls situated at $y = \pm h$ with velocities $\mathbf{v} = \pm U\hat{\mathbf{x}}$ and there is no imposed pressure gradient. Integrating equation (6) with $dP_0/dx = 0$ from the bottom wall where $u = v = 0$ to y gives

$$\nu \frac{d\overline{U}}{dy} - \overline{uv} = \tau_w, \tag{8}$$

where $\tau_w = \nu \, d\overline{U}/dy|_h$ is the stress on the bottom wall. Equation (8) states that the mean *stress* on the fluid layer below y is constant across the channel in Couette flow. The first term on the left hand-side of (8), $\nu \, d\overline{U}/dy$, is the mean viscous stress from the fluid above level y onto the fluid layer below y. The second term, $-\overline{uv}$, is the *Reynolds stress* that arises from the net vertical transport of streamwise momentum by the fluctuations, again from the fluid above y into the fluid layer below y. The two stresses add up to a constant total stress for Couette flow, τ_w. Note that these are all *kinematic* stresses, stress divided by fluid density ρ, so the actual total stress is $\rho\tau_w$.

[1] This is a reasonable assumption, verified experimentally and numerically. However, our recent discoveries of time-periodic solutions in plane Couette flow [4], [7], [15] show that is is possible to have time-dependent averages, in general.

2 Laminar and Inertial Scalings of the Drag τ_w

First we look at Couette flow, as described by equation (8). We try to find appropriate scalings for the two terms on the left hand side using the obvious scalings for velocity and length, U and h, the half-wall velocity difference and the half-channel height, respectively. We can scale the viscous and Reynolds stress terms in (8) as

$$\nu \frac{d\overline{U}}{dy} \sim \frac{\nu U}{h}, \qquad -\overline{uv} \sim U^2, \tag{9}$$

away from the walls, yielding the total stress as

$$\tau_w \sim \frac{\nu U}{h} + U^2 = \frac{\nu U}{h}\left(1 + \frac{Uh}{\nu}\right) = \frac{\nu U}{h}(1+R) \tag{10}$$

where $R = Uh/\nu$ is the Reynolds number for Couette flow. This correctly suggests that for $R \ll 1$, the viscous stress $\nu d\overline{U}/dy$ dominates and we have *laminar scaling*

$$\tau_w \sim \frac{\nu U}{h} \quad \Rightarrow \quad \frac{\tau_w}{U^2} \sim \frac{1}{R}, \tag{11}$$

where τ_w/U^2 is a *friction factor*, as in the Moody diagram of Lecture 1. For $R \gg 1$, the Reynolds stress $-\overline{uv}$ dominates and we have *inertial scaling*

$$\tau_w \sim U^2 \quad \Rightarrow \quad \frac{\tau_w}{U^2} \sim 1. \tag{12}$$

This corresponds to the $R \gg 1$ portion of the Moody diagram labeled *'complete turbulence,'* where the friction factor τ_w/U^2 is roughly independent of Reynolds number R and would correspond to drag being dominated by the Reynolds stress $-\overline{uv}$ scaling as U^2.

Next we look at the scalings for channel flow. Now our equation is (6) with (7),

$$\nu \frac{d^2\overline{U}}{dy^2} - \frac{d\overline{uv}}{dy} = -\frac{\tau_w}{h}. \tag{13}$$

This equation states that the mean *force* on a fluid layer between y and $y + dy$ is a negative constant in channel flow to balance the positive pressure force $-dP_0/dx = \tau_w/h$. Scaling (13) as in (9)

$$\nu \frac{d^2\overline{U}}{dy^2} \sim -\frac{\nu U}{h^2}, \qquad \frac{d\overline{uv}}{dy} \sim \frac{U^2}{h}, \tag{14}$$

away from the walls, yields

$$\frac{\tau_w}{h} \sim \nu \frac{U}{h^2} + \frac{U^2}{h} \sim \nu \frac{U}{h^2}(1+R). \tag{15}$$

This leads to the same results as in Couette flow, namely that the drag at the wall τ_w has the laminar scaling $\tau_w \sim \nu U/h$, corresponding to a friction factor $\tau_w/U^2 \sim 1/R$ for low R (11) and the inertial scaling $\tau_w \sim U^2$ for large R (12). The drag at the wall is the pressure gradient times the half-height, $\tau_w = -h\, dP_0/dx$, as derived in (7).

3 Energy Dissipation Rate $\mathcal{E} = \tau_w U/h$

We now look at the the Kinetic Energy (KE) budget for the total flow (mean plus fluctuation). The KE equation is obtained by the scalar product between the velocity $\mathbf{v}$ and the NS equation (2), yielding

$$\frac{\partial}{\partial t}\left(\frac{1}{2}|\mathbf{v}|^2\right) + \mathbf{v}\cdot\nabla\left(\frac{1}{2}|\mathbf{v}|^2\right) + \mathbf{v}\cdot\nabla p = -\frac{dP_0}{dx}\hat{\mathbf{x}}\cdot\mathbf{v} + \nu\,\mathbf{v}\cdot\nabla^2\mathbf{v}. \tag{16}$$

Equation (16) is then averaged over the whole domain V, that is $V^{-1}\int_V(\cdots)dV$ as a limit process in the homogeneous directions x and z, as in (1), or with periodic boundary conditions as in numerical simulations. The advection and flow-induced pressure terms vanish upon integration since $\nabla\cdot\mathbf{v} = 0$ implies that $\mathbf{v}\cdot\nabla(*) = \nabla\cdot(\mathbf{v}(*))$, then the integral of these divergences become surface integrals, by the divergence theorem, and the boundary integrals vanish because of no flow through the walls and/or periodic boundary conditions. The viscous term $\mathbf{v}\cdot\nabla^2\mathbf{v} = \nabla^2|\mathbf{v}|^2/2 - \nabla\mathbf{v} : \nabla\mathbf{v}^T$ where $\mathbf{A} : \mathbf{B} \equiv A_{ij}B_{ji} = \text{trace}(\mathbf{A}\cdot\mathbf{B})$, so $\nabla\mathbf{v} : \nabla\mathbf{v}^T = (\partial_i v_j)(\partial_i v_j) \equiv |\nabla\mathbf{v}|^2 \geq 0$ is positive definite.

For channel flow in statistically steady state, we obtain

$$\underbrace{-\frac{dP_0}{dx}\,\frac{1}{V}\int_V \hat{\mathbf{x}}\cdot\mathbf{v}dV}_{\textit{energy input rate}} = \underbrace{\frac{\nu}{V}\int_V |\nabla\mathbf{v}|^2 dV}_{\textit{energy dissipation rate}} = \mathcal{E} \geq 0, \tag{17}$$

since $\mathbf{v} = 0$ at the walls. This relation can be written

$$\frac{\tau_w}{h}\,U = \mathcal{E} \tag{18}$$

where $\tau_w/h = -dP_0/dx$, from (7), is the force per unit mass with τ_w the stress at the wall, $U = V^{-1}\int_V \hat{\mathbf{x}}\cdot\mathbf{v}\,dV$ is the bulk velocity and $\mathcal{E}$ is the energy dissipation rate per unit mass.

For plane Couette flow driven by the motion of the walls, with $dP_0/dx = 0$, the viscous term $\mathbf{v}\cdot\nabla^2\mathbf{v} = \nabla^2|\mathbf{v}|^2/2 - \nabla\mathbf{v} : \nabla\mathbf{v}^T$ provides both the energy input and output. The KE equation reads

$$\frac{\nu}{V}\int_V \nabla\cdot\nabla\left(\frac{1}{2}|\mathbf{v}|^2\right)\,dV = \frac{\nu}{V}\int_V |\nabla\mathbf{v}|^2 dV \geq 0. \tag{19}$$

The divergence theorem applied to the left hand side yields

$$\begin{aligned}\frac{\nu}{V}\int_V \nabla\cdot\nabla\left(\frac{|\mathbf{v}|^2}{2}\right)dV &= \frac{\nu}{V}\int_{y=h}\hat{\mathbf{y}}\cdot\nabla\left(\frac{|\mathbf{v}|^2}{2}\right)dA - \frac{\nu}{V}\int_{y=-h}\hat{\mathbf{y}}\cdot\nabla\left(\frac{|\mathbf{v}|^2}{2}\right)dA \\ &= \frac{\nu}{V}\int_{y=h} u\frac{\partial u}{\partial y}dA - \frac{\nu}{V}\int_{y=-h} u\frac{\partial u}{\partial y}\,dA = \frac{\tau_w}{h}\,U\end{aligned} \tag{20}$$

since $u = U$ on the top wall at $y = h$ and $u = -U$ at $y = -h$ with $v = w = 0$ on both walls, and $\tau_w = \nu A^{-1}\int_{y=-h}\frac{\partial u}{\partial y}\,dA = \nu A^{-1}\int_{y=h}\frac{\partial u}{\partial y}\,dA$ is the mean stress on the bottom wall with surface area A, equal to the mean stress from the top wall onto the fluid and $V = 2Ah$. These surface integrals may have to be interpreted as limits as $A \to \infty$, as in (1). Using (20), the KE equation for Couette flow (19) can also be written in the compact form (18),

$$\frac{\tau_w}{h}\,U = \mathcal{E} \tag{21}$$

but this U is the half wall velocity difference in plane Couette flow, instead of the bulk velocity in (18), and h is the half channel height in both cases.

The energy input/dissipation balance (18), (21), for pipe, channel and plane Couette flow, shows that there is a direct relationship between the scaling of the wall stress τ_w (the *drag*) and the scaling of the energy dissipation rate $\mathcal{E}$. For laminar scaling (11), (15) for 'small' Reynolds number $R = Uh/\nu$ (meaning smaller than a few 100's for the typical definitions of R in shear flows)

$$\tau_w \sim \frac{\nu U}{h} \quad \Leftrightarrow \quad \mathcal{E} \sim \nu \frac{U^2}{h^2}, \tag{22}$$

while the inertial scaling (12), (15) for large $R = Uh/\nu$,

$$\tau_w \sim U^2 \quad \Leftrightarrow \quad \mathcal{E} \sim \frac{U^3}{h}. \tag{23}$$

The remarkable fact about the inertial scaling (23) is that the wall stress

$$\tau_w = \nu \left. \frac{d\overline{U}}{dy} \right|_{wall} \tag{24}$$

which is necessarily transmitted from the fluid to the wall by viscosity, and the energy dissipation rate (17), (29)

$$\mathcal{E} = \nu \left\langle |\nabla \mathbf{v}|^2 \right\rangle = 2\,\nu \,\langle \mathbf{S} : \mathbf{S} \rangle, \tag{25}$$

where the brackets $\langle\ \rangle$ denote volume average, which also arises from viscosity, would both be *independent* of ν for sufficiently large R according to (23). This requires boundary layers of thickness $\sim \nu/U$ on average, much smaller than a stagnation point boundary layer thickness $\sim \sqrt{\nu/S}$, where S is a strain rate. The $\sqrt{\nu/S}$ boundary layer is to satisfy no-slip at $y = 0$ for the outer stagnation point flow $\mathbf{v} = (Sx, -Sy, 0)$ in incompressible flow (*e.g.* [1, §2.5]). It is not clear if and how one can obtain a ν/U boundary layer on a smooth wall in incompressible flow.

Note on the energy dissipation rate

The expression for the energy dissipation rate $\mathcal{E}$ in (17) is not correct in general. The viscous term $\nu\nabla^2\mathbf{v}$ in the Navier-Stokes equation (2) arises from the divergence of the stress tensor $\mathbf{T} = \nu(\nabla\mathbf{v} + \nabla\mathbf{v}^T) = 2\nu\mathbf{S}$, where $\mathbf{S}$ is the strain rate tensor, then $\nabla \cdot \mathbf{T} = \nu\nabla^2\mathbf{v}$ because $\nabla \cdot (\nabla\mathbf{v})^T = \nabla(\nabla \cdot \mathbf{v}) = 0$, since $\nabla \cdot \mathbf{v} = 0$. In the kinetic energy equation, then,

$$\nu\,\mathbf{v} \cdot \nabla^2\mathbf{v} = \nu\,\nabla \cdot ((\nabla\mathbf{v}) \cdot \mathbf{v}) - \nu\,\nabla\mathbf{v} : \nabla\mathbf{v}^T \tag{26}$$

is equal to

$$\mathbf{v} \cdot (\nabla \cdot \mathbf{T}) = \nabla \cdot (\mathbf{T} \cdot \mathbf{v}) - \mathbf{T} : \mathbf{S}, \tag{27}$$

where symmetry of the stress tensor $\mathbf{T} = \mathbf{T}^T$ has been used in (27) to obtain $\mathbf{T} : \nabla\mathbf{v}^T = T_{ij}\partial_i v_j = T_{ij}(\partial_i v_j + \partial_j v_i)/2 = T_{ij}S_{ji} = \mathbf{T} : \mathbf{S}$. Integration over the domain shows that the second form (27) yields the proper energy input rate on the boundaries, since, by the divergence theorem

$$\int_V \nabla \cdot (\mathbf{T} \cdot \mathbf{v})\ dV = \int_{\partial V} \mathbf{f} \cdot \mathbf{v}\, dS \tag{28}$$

where $\mathbf{f} = \mathbf{n}\cdot\mathbf{T}$ is the stress on the surface boundary ∂V of the volume V, with unit outward normal $\mathbf{n}$, by definition of the stress tensor $\mathbf{T}$. Hence the energy dissipation rate per unit mass arises from the 2nd term on the right hand side of (27)

$$\mathcal{E} = \frac{1}{V}\int_V \mathbf{T} : \mathbf{S}\, dV = \frac{1}{V}\int_V 2\nu\, \mathbf{S} : \mathbf{S}\, dV \geq 0, \tag{29}$$

not from the 2nd term of (26), $\nu\,\nabla\mathbf{v} : \nabla\mathbf{v}^T$.

Subtracting (26) from (27) and canceling out factors of ν shows that the expressions $\nabla\mathbf{v} : \nabla\mathbf{v}^T$ and $2\,\mathbf{S} : \mathbf{S}$ differ by a divergence,

$$2\,\mathbf{S} : \mathbf{S} = \nabla\mathbf{v} : \nabla\mathbf{v}^T + \nabla\cdot(\mathbf{v}\cdot\nabla\mathbf{v}). \tag{30}$$

The divergence term on the right hand side integrates to

$$\int_V \nabla\cdot(\mathbf{v}\cdot\nabla\mathbf{v}) dV = \int_{\partial V} \mathbf{v}\cdot(\nabla\mathbf{v})\cdot\mathbf{n}\, dS \tag{31}$$

and that boundary term vanishes for periodic boundary conditions, or for $\mathbf{v} = 0$ on ∂V, or for *flat* boundaries for which $\mathbf{n}$ is constant so $(\nabla\mathbf{v})\cdot\mathbf{n} = \nabla(\mathbf{v}\cdot\mathbf{n})$ and there is no flow through the walls, $\mathbf{v}\cdot\mathbf{n} = 0$, so $\mathbf{v}\cdot(\nabla\mathbf{v})\cdot\mathbf{n} = \mathbf{v}\cdot\nabla(\mathbf{v}\cdot\mathbf{n}) = 0$ on ∂V.

Thus, the expressions (17) and (29) for $\mathcal{E}$ are equal for pipe and plane channel and Couette flows, but they differ for cylindrical Couette flow, for instance. The difference is most striking for rigid body rotation for which $\mathbf{v} = (-\Omega y, \Omega x, 0)$ and $\nabla\mathbf{v} : \nabla\mathbf{v}^T = 2\Omega^2$ but $2\,\mathbf{S} : \mathbf{S} = 0$.

4 Kolmogorov spectrum and Energy cascade

The study of turbulence spawned a new branch in the 1930's when researchers went beyond the study of mean flows $\overline{U}(y)$ and started focusing on the turbulent fluctuations. Experiments measured *two-point correlations* $\langle \mathbf{v}(\mathbf{r}')\mathbf{v}(\mathbf{r})\rangle$, where the average denoted by the brackets $\langle\ \rangle$ is typically a time average in practice or an ensemble average in theory. G.I. Taylor introduced the concept of *homogeneous turbulence* where those time or ensemble averages are assumed to depend only on the two-point separation $\mathbf{s} = \mathbf{r}' - \mathbf{r}$ but not on location, *e.g.*

$$\langle \mathbf{v}(\mathbf{r}')\mathbf{v}(\mathbf{r})\rangle = \mathbf{R}(\mathbf{s}), \tag{32}$$

that is $\langle v_i(\mathbf{r}')v_j(\mathbf{r})\rangle = R_{ij}(\mathbf{s})$ in cartesian index notation. The possible time dependence of these correlations will be kept implicit in this section. For fully developed turbulent flow in a pipe or channel at sufficiently high Reynolds number, the turbulence is observed to be homogeneous in the azimuthal (or spanwise) and streamwise directions, sufficiently far from the entrance and exit to the pipe, and from the sides in channels. The statistics are strongly dependent on the distance to the wall, near the walls, but become approximately homogeneous sufficiently far from the wall. Sufficiently far is best measured in *wall units* (typically denoted with a '+' and sometimes called 'plus-units')

$$\delta^+ = \frac{\nu}{\sqrt{\tau_w}} \tag{33}$$

where $\tau_w = \nu \, d\overline{U}/dy|_w$ is the (kinematic) stress at the wall (24) which has units of velocity squared. For laminar scaling (11),

$$\tau_w \sim \nu \frac{U}{h} \quad \Rightarrow \quad \delta^+ \sim \sqrt{\frac{\nu h}{U}}, \tag{34}$$

but for inertial scaling (12), the wall unit would be much smaller

$$\tau_w \sim U^2 \quad \Rightarrow \quad \delta^+ \sim \frac{\nu}{U}. \tag{35}$$

In practice, 'sufficiently far' means distances from the wall greater than about $50\,\delta^+$.

The theory of *homogeneous turbulence*, where the statistics are invariant under translations, quickly specialized to the study of *isotropic turbulence* where the statistics are also independent of rotations and reflections. For the two-point correlation (32), that involves vector quantities $\mathbf{v}$ and introduces the special direction $\mathbf{s} = \mathbf{r}' - \mathbf{r}$, this implies

$$\langle \mathbf{v}(\mathbf{r}')\mathbf{v}(\mathbf{r})\rangle = u_{rms}^2 \Big(F(s)\,\hat{\mathbf{s}}\hat{\mathbf{s}} + G(s)\,(\mathbf{I} - \hat{\mathbf{s}}\hat{\mathbf{s}}) \Big) \tag{36}$$

in dyadic notation, where s is the magnitude and $\hat{\mathbf{s}}$ the direction of $\mathbf{s} = s\,\hat{\mathbf{s}}$ and $\mathbf{I}$ is the identity tensor, or

$$\langle v_i(\mathbf{r}')v_j(\mathbf{r})\rangle = u_{rms}^2 \left(F(s)\frac{s_i s_j}{s^2} + G(s)\left(\delta_{ij} - \frac{s_i s_j}{s^2}\right)\right) \tag{37}$$

in cartesian index notation, where δ_{ij} is the Kronecker delta and s_i are the cartesian components of $\mathbf{s} = \mathbf{r}' - \mathbf{r}$. The root mean square velocity is $u_{rms}^2 = \langle \mathbf{v} \cdot \mathbf{v} \rangle / 3$. Equation (36) $\equiv$ (37), says that the correlation depends only on the distance s between the two points and whether the velocity components are parallel or perpendicular to $\mathbf{s}$. Incompressibility, $\nabla \cdot \mathbf{v} = 0$ applied to (36), (37), yields

$$s\frac{dF}{ds} + 2(F - G) = 0 \tag{38}$$

relating $G(s)$ to $F(s)$, so the two-point velocity correlation tensor (32) is fully determined by the *longitudinal auto-correlation function*,

$$F(s) \equiv \frac{\langle u_1(\mathbf{r} + s\mathbf{e}_1)\, u_1(\mathbf{r})\rangle}{u_{rms}^2} \tag{39}$$

in homogeneous isotropic turbulence. The reader should consult the books by Pope [13] or Sagaut and Cambon [14] for further information about homogeneous turbulence.

In homogeneous isotropic turbulence, the flow is assumed to take place in an infinite 3D euclidean space, maintained by a statistically steady and isotropic force but there are no mean flows, no walls and no drag. Thus the relationship (18), between the wall drag τ_w and the energy dissipation rate $\mathcal{E}$ is lost, although there is of course a similar relationship involving energy input by the statistically homogeneous, isotropic force.

In the Kolmogorov theory of isotropic turbulence, the energy dissipation rate $\mathcal{E}$ is the dominant quantity, it is the *energy cascade rate,* with energy input at the forcing scale ℓ_I cascading to ever smaller scales by nonlinear distortion down to sufficiently small scales where the energy is finally dissipated by viscosity. Kolmogorov estimated those small dissipation length scales by dimensional analysis based on $\mathcal{E}$ that has units L^2T^{-3} and viscosity ν with units L^2T^{-1}, so the *Kolmogorov dissipation length scale*, is

$$\ell_K \sim \left(\frac{\nu^3}{\mathcal{E}}\right)^{1/4}. \tag{40}$$

Dissipation on those scales would be achieved by turbulent velocity fluctuations of magnitude

$$u_K \sim (\nu\mathcal{E})^{1/4}, \tag{41}$$

with

$$\mathcal{E} \sim \nu \frac{u_K^2}{\ell_K^2} \sim \frac{u_{rms}^3}{\ell_I} \tag{42}$$

independent of ν, where $3u_{rms}^2 = \langle \mathbf{v} \cdot \mathbf{v} \rangle$ and ℓ_I is the energy input scale.

Kolmogorov's theory also predicts a scaling for the *Energy spectrum*

$$E(k) \sim \mathcal{E}^{2/3}\, k^{-5/3} \tag{43}$$

where $\int_0^\infty E(k)dk = \langle \mathbf{v} \cdot \mathbf{v} \rangle/2 = 3u_{rms}^2/2$ and $E(k)dk$ is the energy per unit mass in the wavenumber band $[k, k + dk]$. In isotropic turbulence, the energy spectrum $E(k)$ fully determines the Fourier transform $\Phi_{ij}(\mathbf{k})$ of the two point correlation tensor $R_{ij}(\mathbf{s})$ defined in (32)

$$\Phi_{ij}(\mathbf{k}) = \frac{E(k)}{4\pi k^2}\left(\delta_{ij} - \frac{k_i k_j}{k^2}\right) \tag{44}$$

where

$$\Phi_{ij}(\mathbf{k}) \equiv \frac{1}{(2\pi)^3}\int_{\mathbb{R}^3} R_{ij}(\mathbf{s})\, e^{-i\mathbf{k}\cdot\mathbf{s}} dV_{\mathbf{s}} \tag{45}$$

$$R_{ij}(\mathbf{s}) = \int_{\mathbb{R}^3} \Phi_{ij}(\mathbf{k})\, e^{i\mathbf{k}\cdot\mathbf{s}} dV_{\mathbf{k}} \tag{46}$$

with $dV_{\mathbf{s}}$ the volume element for $\mathbf{s} \in \mathbb{R}^3$ and $dV_{\mathbf{k}}$ the volume element for $\mathbf{k} \in \mathbb{R}^3$, so $dV_{\mathbf{s}} = ds_1 ds_2 ds_3$ and $dV_{\mathbf{k}} = dk_1 dk_2 dk_3$ in cartesian coordinates and $k = |\mathbf{k}|$. The energy spectrum $E(k)$ can be related to $F(s)$, the longitudinal auto-correlation (39), although the relationship is non-trivial

$$u_{rms}^2 F(s) = 2\int_0^\infty E(k)\left(\frac{\sin(ks)}{(ks)^3} - \frac{\cos(ks)}{(ks)^2}\right)\, dk. \tag{47}$$

A simpler relationship can be derived from (37), (44) and (46)

$$R_{ii}(\mathbf{s}) = u_{rms}^2\left(F(s) + 2G(s)\right) = 2\int_{\mathbb{R}^3} \frac{E(k)}{4\pi k^2}\, e^{i\mathbf{k}\cdot\mathbf{s}} dV_{\mathbf{k}} = 2\int_0^\infty E(k)\frac{\sin(ks)}{ks} dk. \tag{48}$$

The Kolmogorov spectrum (43) occurs in the *inertial range* $\ell_I \gg k^{-1} \gg \ell_K$ where the wavelengths k^{-1} are much smaller than the energy input scale ℓ_I but much larger than the

Kolmogorov dissipation scale ℓ_K. Defining the Reynolds number for homogeneous isotropic turbulence as

$$R = \frac{u_{rms}\,\ell_I}{\nu} = \frac{\mathcal{E}^{1/3}\ell_I^{4/3}}{\nu}. \tag{49}$$

Using (40) then gives

$$\frac{\ell_I}{\ell_K} = \frac{\mathcal{E}^{1/4}\ell_I}{\nu^{3/4}} \equiv R^{3/4}. \tag{50}$$

providing a rule of thumb, first proposed by S.A. Orszag, that the numerical resolution scales like $N \sim R^{9/4}$ for direct numerical simulation of 3D isotropic turbulence, where N is the total number of Fourier modes (or grid points) required to resolve the turbulent flow. A sketch of the energy spectrum with the inertial range (43) and the dissipation range for $k \gtrsim (\ell_K)^{-1}$ is shown in figure 1.

Since $\mathcal{E}$ is assumed to be independent of ν for sufficiently large Reynolds numbers, with $\mathcal{E} \sim u_{rms}^3/\ell_I$, eqn. (42) as in the inertial scaling (23), the Kolmogorov dissipation length (40) scales like $\nu^{3/4}$, smaller than the classic $\sqrt{\nu}$ boundary layer scaling, but *larger* than the inertial boundary layer scaling ν/U in turbulent channels and pipes (35). Thus, although the inertial scaling $\mathcal{E} \sim U^3/h$ in channels and pipes (23) and the Kolmogorov scaling $\mathcal{E} \sim u_{rms}^3/\ell_I$ are similar, there is a noteworthy difference between the two: the Kolmogorov scaling is achieved by small dissipation scales $\sim \nu^{3/4}$ throughout the volume of fluid, while the inertial scaling in shear flows over smooth walls requires small boundary layers $\sim \nu/U$ to transfer the momentum between the fluid and the walls, so $\tau_w = \nu\, d\overline{U}/dy|_w \sim U^2$, and would achieve ν-independent $\mathcal{E}$ through dissipation in those small boundary layers.

The Kolmogorov picture of turbulence is a *cascade of energy to smaller and smaller scales,* from the energy input scale ℓ_I down to the Kolmogorov dissipation scale $\sim \nu^{3/4}$, together with a *decoherence of the motions at different scales so that small scales are nearly isotropic.* In contrast, the inertial scaling (23) in shear flows appears to call for a *coherent transport of momentum over the height of the channel,* from one wall to the other wall in plane Couette flow for instance, transporting U over distance h to sustain boundary layers of thickness $\sim \nu/U$, on average. It is also known that in turbulent shear flows, the turbulence energy input occurs at small scales near the wall, not on the large scale of the channel.

5 ***Turbulent*** Kinetic Energy

In section 3, we derived a basic relation between the *drag* τ_w and the total *energy dissipation rate* $\mathcal{E}$. Here, we derive an equation for the fluctuations from the mean flow in turbulent channel flows. We return to the Reynolds decomposition $\mathbf{v} = \bar{\mathbf{v}} + \mathbf{v}'$ with $\mathbf{v} = \overline{U}(y,t)\hat{\mathbf{x}}$ in channels (4) and $\mathbf{v}' = (u, v, w)$ in cartesian coordinates. Substituting $\mathbf{v} = \overline{U}(y,t)\hat{\mathbf{x}} + \mathbf{v}'$ in the Navier-Stokes equations (3) and subtracting the mean flow equation (5) yields the equation for the turbulent fluctuations

$$\frac{\partial \mathbf{v}'}{\partial t} + \overline{U}\frac{\partial \mathbf{v}'}{\partial x} + v\frac{\partial \overline{U}}{\partial y}\hat{\mathbf{x}} + \nabla\cdot(\mathbf{v}'\mathbf{v}') - \frac{\partial \overline{uv}}{\partial y}\hat{\mathbf{x}} = -\nabla p' + \nu\nabla^2\mathbf{v}', \tag{51}$$

where $\overline{U}$ and $\overline{uv}$ are functions of y and t in general. Dotting this equation (51) with $\mathbf{v}'$ and averaging over the whole volume using the divergence theorem, incompressibility $\nabla\cdot\mathbf{v}' = 0$

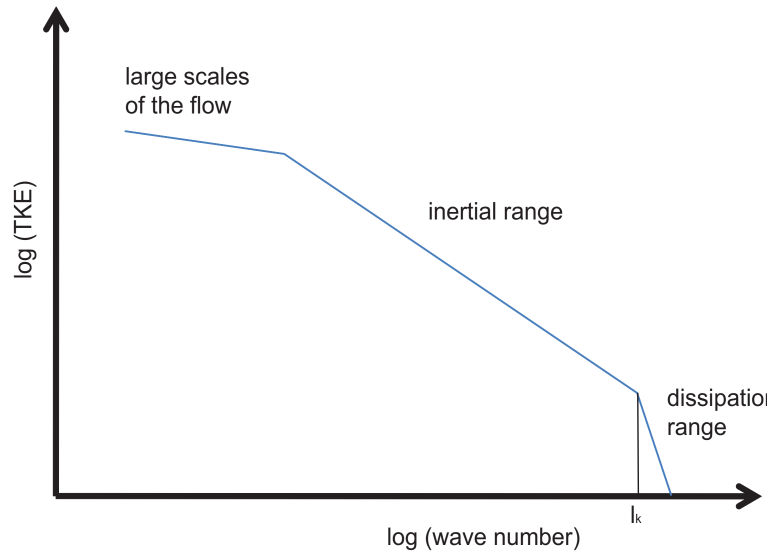

Figure 1: Sketch of the Turbulent Kinetic Energy spectrum $E(k)$ where k is wavenumber and $E(k)dk$ is the kinetic energy in the wavenumber band $[k, k + dk]$, according to the Kolmogorov scaling theory. Energy enters the flow at low wavenumbers (large scales) and cascades through the inertial range (43) to the dissipation scales (40) where it is dissipated.

and the boundary conditions, yields the kinetic energy of the fluctuations

$$\frac{d}{dt}\left\langle \frac{|\mathbf{v}'|^2}{2} \right\rangle = \underbrace{\left\langle (-uv)\frac{\partial \overline{U}}{\partial y} \right\rangle}_{Production} - \underbrace{\nu \left\langle |\nabla \mathbf{v}'|^2 \right\rangle}_{Dissipation\,\geq 0} \tag{52}$$

where the brackets $\langle \cdots \rangle \equiv V^{-1} \int_V \cdots dV$ denote a volume average, $|\nabla \mathbf{v}'|^2 = \partial_i u_j \partial_i u_j$ in cartesian index notation (but recall (30) in general), and the $\mathbf{v}' \cdot \hat{\mathbf{x}}\, \partial_y \overline{uv} = u\, \partial_y \overline{uv}$ term vanishes upon averaging over horizontal planes since $\bar{u} = 0$. This fluctuating kinetic energy equation should be contrasted with the mean kinetic energy equation obtained by multiplying the mean flow equation (5) by $\overline{U}$ and averaging over the volume to obtain

$$\frac{d}{dt}\left\langle \frac{\overline{U}^2}{2} \right\rangle = \frac{\tau_w}{h} U - \left\langle (-uv)\frac{\partial \overline{U}}{\partial y} \right\rangle - \nu \left\langle \left(\frac{\partial \overline{U}}{\partial y} \right)^2 \right\rangle \tag{53}$$

where $\tau_w U/h$ arises as in section 3 and U is the bulk velocity in channel flow, but the half wall velocity difference in plane Couette flow, in either case $\tau_w U/h$ is the energy input per unit mass from the pressure gradient or the wall drag. Evidently, the 'production' term in (52) appears with the opposite sign in the mean flow kinetic energy (53) and is a transfer term extracting energy from the mean flow $\overline{U}$ to feed the fluctuations $\mathbf{v}'$. Adding up (52) and (53) leads back to the total kinetic energy equation of section 3, that reduces to (18), (21) for statistically steady flow. For statistically steady state, the fluctuation kinetic energy equation (52) becomes

$$\frac{1}{2h}\int_{-h}^{h} (-\overline{uv})\frac{d\overline{U}}{dy}\, dy = \nu \left\langle |\nabla \mathbf{v}'|^2 \right\rangle \geq 0. \tag{54}$$

where the brackets $\langle \cdots \rangle$ denote a volume average and the overline $\overline{(\cdots)}$ denotes a horizontal average. This equation states that $(-\overline{uv})$ and $d\overline{U}/dy$ must have the same sign, on average, in order to sustain the turbulent fluctuations.

6 Upper bound on drag and dissipation

Malkus [10] proposed a theory of turbulent convection and of shear turbulence where he invoked concepts of *marginal stability*, *existence of a smallest scale* and *maximization of heat flux or momentum transport*. This led Howard [5, 6] and Busse [2] to derive bounds on heat flux in convection and momentum transport in shear flow.

For plane Couette flow, the idea is to look for the field $\mathbf{v}' = (u, v, w)$ that maximizes the drag τ_w (8) for a given U subject to various constraints including the boundary conditions $\overline{U} = \pm U$ and $\mathbf{v}' = 0$ at $y = \pm h$, incompressibility $\nabla \cdot \mathbf{v}' = 0$ and the fluctuation energy equation (54). The expression (8) for the total stress,

$$\tau_w = \nu \frac{d\overline{U}}{dy} - \overline{uv} \tag{55}$$

can be averaged over the fluid layer $-h \leq y \leq h$ to obtain

$$\tau_w = \frac{\nu U}{h} - \langle uv \rangle \tag{56}$$

since $\overline{U}(y = \pm h) = \pm U$ and τ_w is constant, and where the brackets $\langle \cdots \rangle$ denote a volume average. While $\nu d\overline{U}/dy$ and $-\overline{uv}$ in (8) are functions of y that add up to the drag τ_w, equation (56) expresses the drag as a the laminar value $\nu U/h$ plus a nonlinear, turbulent contribution $-\langle uv \rangle$. Normalizing (56) by U^2 yields

$$\frac{\tau_w}{U^2} = \frac{1}{R} - \frac{\langle uv \rangle}{U^2} \tag{57}$$

showing the $1/R$ scaling of the *friction factor* τ_w/U^2 (recall the Moody diagram of Lecture 1) in laminar flow where $\langle uv \rangle = 0$.

Eliminating τ_w between (55) and (56) gives $\nu d\overline{U}/dy = \nu U/h + \overline{uv} - \langle uv \rangle$ which we can then use to eliminate the mean shear $d\overline{U}/dy$ in the fluctuation energy equation (54) to obtain the constraint

$$-\langle uv \rangle \frac{U}{h} + \frac{1}{\nu}\Big(\langle \overline{uv} \rangle^2 - \langle \overline{uv}^2 \rangle \Big) = \nu \, \big\langle |\nabla \mathbf{v}'|^2 \big\rangle \tag{58}$$

since $\langle \overline{uv} \rangle = \langle uv \rangle$. The $1/\nu$ term on the left hand side is negative definite and can be written $\langle \overline{uv} \rangle^2 - \langle \overline{uv}^2 \rangle = -\big\langle (\overline{uv} - \langle uv \rangle)^2 \big\rangle$, thus (58) shows that we need $\langle uv \rangle < 0$ to sustain turbulent fluctuations. In the standard non-dimensionalization of velocities by U and lengths by h, the energy constraint (58) reads

$$-\langle uv \rangle = R \left\langle (\overline{uv} - \langle uv \rangle)^2 \right\rangle + \frac{1}{R} \left\langle |\nabla \mathbf{v}'|^2 \right\rangle \geq 0 \tag{59}$$

with the Reynolds number $R = Uh/\nu$. The upper bound problem is then to find the field $\mathbf{v}' = (u, v, w)$, with $\nabla \cdot \mathbf{v}' = 0$ and $\mathbf{v}' = 0$ on the boundaries, that maximizes $-\langle uv \rangle$

subject to the energy constraint (59) for fixed Reynolds number R. Busse [2] uses a different convention and normalization and actually solves for the minimum Reynolds number for a given momentum transport $\mu \equiv -\langle uv \rangle \geq 0$. Busse argues that the optimum solution is streamwise x independent and solves the problem approximately using so-called 'multi-α solutions' that provide the interesting multi-scale optimum field shown in figure 2.

Doering and Constantin [3] have developed a different 'background flow' approach to bounds on flow quantities and Kerswell [8] has shown the relationship between the Howard-Busse and the Doering-Constantin approaches. Plasting and Kerswell [12] obtain the bound

$$\frac{\tau_w}{h} U = \mathcal{E} < 0.034 \frac{U^3}{h} \tag{60}$$

for plane Couette flow in the limit $R = Uh/\nu \to \infty$, where U is the half wall velocity difference and h is the half distance between the walls.

7 Mean flow phenomenology

We now consider some features of a fully turbulent (*i.e.* R well above transition) shear flow near a solid boundary. The following arguments are discussed in Kundu and Cohen [9, p. 570] and Pope [13, Chap. 7]. We consider the case where the boundary roughness is small, so it does not affect the flow. In this section we think of y as the distance to the (bottom) wall, not to the centerline as earlier in these notes.

Prandtl (1925) proposed that at sufficiently large Reynolds number $R = Uh/\nu \gg 1$, in a region sufficiently close to the wall $y \ll h$, the mean velocity profile $\overline{U}(y)$ should have a universal form, independent of h. This is in the spirit of the inertial scaling discussed earlier. Indeed, the drag at the wall $\tau_w = \nu d\overline{U}/dy|_{wall}$ is a function of the bulk or wall velocity U, the half channel height h and the kinematic viscosity ν, in general, but in non-dimensional form this reads

$$\tau_w = U^2 f(R) \tag{61}$$

for some function $f(R)$ where $R = Uh/\nu$ is the Reynolds number. This should be obvious enough but follows more generally from the 'Buckingham Pi theorem'. Recall that the plot of τ_w/U^2 as a function of R is the *Moody diagram* shown for pipe flow in lecture 1. The assumption of *inertial scaling* is that $f(R) \to$ *constant* as $R \to \infty$, an asymptotic regime labeled *'complete turbulence'* on the Moody diagram. This is equivalent to saying that for sufficiently large R, the drag at the wall τ_w is independent of the size of the channel h, which then implies that it is also independent of ν and can only be proportional to U^2, by dimensional analysis. It is also equivalent to making the assumption that τ_w is independent of ν as $R \to \infty$, in which case it is also independent of h by dimensional analysis.

The equation for the mean flow in plane Couette (8) can be written

$$\frac{d\overline{U}}{dy} = \frac{u_\tau^2}{\nu} + \frac{\overline{uv}}{\nu} \tag{62}$$

where it will be convenient to write the drag as $\tau_w = u_\tau^2$. In channel flow, integrating (6) over y with (7) gives

$$\frac{d\overline{U}}{dy} = \frac{u_\tau^2}{\nu}\left(1 - \frac{y}{h}\right) + \frac{\overline{uv}}{\nu}. \tag{63}$$

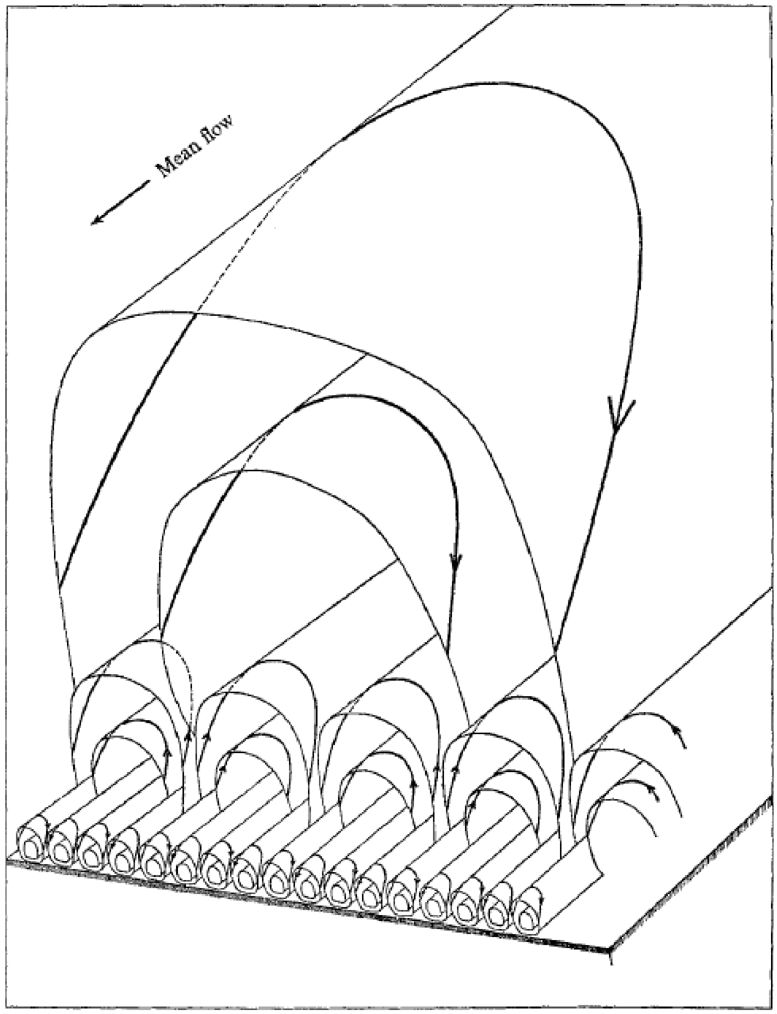

Figure 2: Qualitative sketch of the boundary-layer region of the vector field yielding maximum transport of momentum, from Busse [2]. The streamlines depict the fiuctuation field $\mathbf{v}' = (u, v, w)$ and show that $\langle uv \rangle < 0$ with $u > 0$ when $v < 0$ and $u < 0$ when $v > 0$.

Recall that we have shifted the y coordinate so that the bottom wall is at $y = 0$ and the channel centerline is at $y = h$ for the purpose of this section, with $\overline{U}(0) = 0$ and $\overline{U}(2h) = 2U$ in plane Couette flow. From the boundary condition and incompressibility, $\nabla \cdot \mathbf{v}' = 0$, we have u, v, $\partial v/\partial y \to 0$ as $y \to 0$ and $d\overline{U}/dy \to u_\tau^2/\nu$ as $y \to 0$ so $\overline{U} \sim u_\tau^2 y/\nu$ in both Couette and channel flow. This suggests introducing the *friction velocity* u_τ (often written u_*) and the *'wall unit'* δ^+,

$$u_\tau \equiv \sqrt{\tau_w}, \qquad \delta^+ \equiv \frac{\nu}{u_\tau}, \tag{64}$$

so that in wall units the asymptotic relation $\overline{U} \sim u_\tau^2 y/\nu$ as $y \to 0$ simply reads

$$\overline{U}^+ \sim y^+ \tag{65}$$

where $\overline{U}^+ = \overline{U}/u_\tau$ and $y^+ = y/\delta^+ = yu_\tau/\nu$. Experiments and simulations show that (65) holds in $0 \leq y^+ \lesssim 5$, a region that is called the *viscous sublayer*. The *friction Reynolds number* is then defined as

$$R_\tau = \frac{u_\tau h}{\nu} = \frac{h}{\delta^+}, \tag{66}$$

and the channel centerline in wall units is at $y^+ = R_\tau$. In laminar flow, $\tau_w \sim \nu U/h$ so $R_\tau \sim \sqrt{R}$ for 'low' R but the inertial scaling $\tau_w \sim U^2$ gives $R_\tau \sim R$ for large R. Pope [13, p. 279] gives the approximation $R_\tau \approx 0.09R^{0.88}$ for turbulent channel flow, but see also (78) and (80) below, for a linear relationship up to a log correction.

These considerations suggest that it may be more appropriate to use u_τ as the characteristic velocity for the mean profile $\overline{U}(y)$ instead of the bulk or wall velocity U and $\delta^+ = \nu/u_\tau$ as the length scale instead of h. So the mean profile $\overline{U}(y)$ depends on u_τ, h, ν and y but by dimensional analysis (and the Pi theorem if necessary) we can write in full generality

$$\frac{d\overline{U}}{dy} = \frac{u_\tau^2}{\nu}\, f(y^+, R_\tau) \tag{67}$$

for some function $f(\cdot,\cdot)$ such that $\lim_{y^+\to 0} f(y^+, R_\tau) = 1$, for any R_τ, as follows from (62) and (63). Note that $y/h = y^+/R_\tau$. Prandtl's *law of the wall* postulates that

$$\lim_{\substack{R_\tau \to \infty \\ y^+\, fixed}} f\left(y^+, R_\tau\right) = \Phi(y^+), \tag{68}$$

and therefore

$$\frac{d\overline{U}}{dy} \approx \frac{u_\tau^2}{\nu}\, \Phi(y^+), \tag{69}$$

for arbitrary but fixed y^+ as $R_\tau \to \infty$, *i.e.* for $y \ll h$ with $R_\tau \gg 1$. Equation (69) can be written $d\overline{U}^+/dy^+ \approx \Phi(y^+)$ and there is indeed good experimental and numerical evidence that the mean profile *scales in wall units*, that is, mean profiles corresponding to different R_τ will 'collapse' onto one another when plotted in wall units, $\overline{U}^+(y^+)$.

The law of the wall (69) implies that $\overline{U}(y)$ is independent of h for $y/h \to 0$. The von Karman *log law* can then be derived by assuming further that $d\overline{U}/dy$ is also independent of ν for sufficiently large y^+, this would require both $\delta^+ \ll y \ll h$, and require $R_\tau = h/\delta^+ \gg 1$. Recall that in the inertial scaling $\tau_w \sim U^2$ is independent of both h and ν. For $d\overline{U}/dy$ in

(69) to be independent of ν requires $\Phi(y^+) \sim 1/y^+$ giving $d\overline{U}/dy \sim u_\tau/y$. This is equivalent to $d\overline{U}^+/dy^+ \sim 1/y^+$ and yields

$$\overline{U}^+ \approx \frac{1}{\kappa}\ln(y^+) + C, \tag{70}$$

where κ is known as the von Karman constant. Experiments and simulations show that $\kappa \approx 0.41$ and $C \approx 5.2$ and the log law (70) holds approximatively in $30\,\delta^+ \lesssim y \lesssim 0.3\,h$. The region $5 \lesssim y^+ \lesssim 30$ where the mean velocity profile transitions from the viscous behavior $\overline{U}^+ \sim y^+$ (65) to the log law (70) is called the *buffer region*.

Von Karman (1930) derived the log law using unsatisfying mixing length arguments and 2D flow considerations, instead of the reasoning presented above. Millikan (1938) provided a more satisfying asymptotic 'overlap' argument (see *e.g.* [9]) that matches Prandtl's law of the wall for $y \ll h$ to the *velocity defect law* that applies for $y \gg \delta^+$. Prandtl's law of the wall (69) can be written

$$\overline{U}(y) \sim u_\tau\, F(y^+), \tag{71}$$

or $\overline{U}^+ \sim F(y^+)$ where $\overline{U}^+ = \overline{U}/u_\tau$, with $F(y^+) \sim y^+$ as $y^+ \to 0$ (65) and $dF/dy^+ = \Phi(y^+)$. This law applies for $y \ll h$ and $R_\tau \gg 1$. There is a similar *velocity defect law* that would apply for $y \gg \delta^+ = \nu/u_\tau$. By dimensional analysis, we can write

$$\overline{U}(y) - \overline{U}_C = u_\tau\, g(\eta, R_\tau) \tag{72}$$

for some function $g(\cdot,\cdot)$, where $\overline{U}_c$ is the centerline velocity and $\eta \equiv y/h$ with $y = 0$ at the wall to the channel center at $y = h$. The *velocity defect law* states that the velocity defect, $\overline{U}(y) - \overline{U}_C$, depends only on u_τ and $\eta = y/h$ but not ν

$$\overline{U}(y) - \overline{U}_C \sim u_\tau\, G(\eta) \tag{73}$$

for $R_\tau \to \infty$ with $\eta = y/h = y^+/R_\tau$ fixed but arbitrary. In the overlap region, $\delta^+ \ll y \ll h$, both equations (71) and (73) should hold and

$$\frac{dU}{dy} \sim \frac{u_\tau^2}{\nu}\frac{dF}{dy^+}, \tag{74}$$

$$\frac{dU}{dy} \sim \frac{u_\tau}{h}\frac{dG}{d\eta}. \tag{75}$$

Multiplying both equations by y/u_τ yields

$$y^+\frac{dF}{dy^+} \sim \eta\frac{dG}{d\eta} \tag{76}$$

since $y^+ = yu_\tau/\nu$ and $\eta = y/h$. Considering $R_\tau \to \infty$ with y^+ fixed but arbitrary gives

$$y^+\frac{dF}{dy^+} \sim \lim_{\eta\to 0}\left(\eta\frac{dG}{d\eta}\right) \equiv \frac{1}{\kappa} \tag{77}$$

where κ is the von Karman constant and this yields the log law (70).

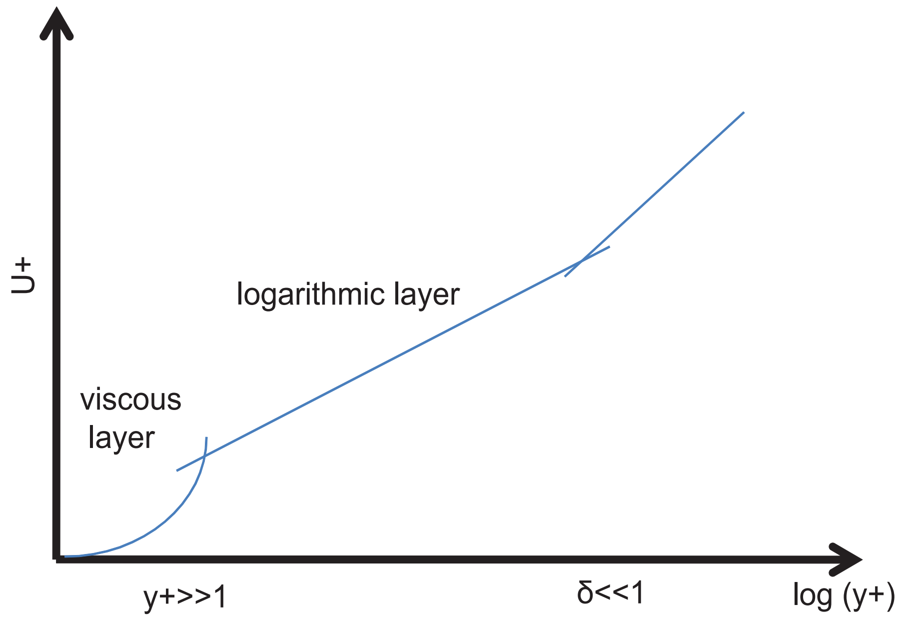

Figure 3: Sketch of the 'Law of the Wall' with its viscous sublayer $\overline{U}^+ \sim y^+$ for $y^+ \lesssim 5$ and the log region $\overline{U}^+ \sim \kappa^{-1} \ln y^+ + C$ for $30 \lesssim y^+ \lesssim 0.3R_\tau$.

The von Karman log law (70) is a good approximation to the mean velocity in most of the channel except for a small viscous region with $y^+ \lesssim 30$, *i.e.* $y \lesssim 30\nu/u_\tau$ and a small deviation near the center of the channel. It can therefore be used to derive an approximation to the friction factor $\tau_w/U^2 = f(R)$ for large R. Integrating the log law (70) from $y^+ \approx 30$ to the centerline at $y^+ = R_\tau \gg 1$ and dividing by the half size of the channel in wall units, R_τ, gives the bulk velocity U as

$$\frac{U}{u_\tau} = \frac{R}{R_\tau} \approx \frac{1}{R_\tau} \int_{30}^{R_\tau} \left(\frac{\ln y^+}{\kappa} + C \right) dy^+ \approx \frac{\ln R_\tau}{\kappa}, \tag{78}$$

for $R_\tau \gg 1$. Defining $u_\tau/U = \sqrt{\lambda}$ where $\lambda = \tau_w/U^2$ is a friction factor, equation (78) gives

$$\frac{1}{\sqrt{\lambda}} = \frac{R}{R_\tau} \approx 2.44 \ln R_\tau = 2.44 \ln \left(R\sqrt{\lambda} \right) \tag{79}$$

for $1/\kappa \approx 2.44$, for channel flow. The friction factor is often defined as $2\tau_w/U^2$ in the literature, so watch out for factors of 2. In any case, the friction factor would not be quite R independent as $R \to \infty$ and there would be a relatively small log correction to that inertial scaling.

For plane Couette flow, we could simply evaluate (70) at the centerline $y = h$ so $y^+ = R_\tau$ where $\overline{U} = U$ (since $\overline{U}(0) = 0$ and $\overline{U}(2h) = 2U$ in the convention of this section) to obtain

$$\frac{U}{\sqrt{\tau_w}} \equiv \frac{U}{u_\tau} \equiv \frac{R}{R_\tau} \approx \frac{1}{\kappa} \ln R_\tau + C \tag{80}$$

and therefore

$$\frac{\tau_w U}{h} = \mathcal{E} \approx \frac{1}{(5.2 + 2.44 \ln R_\tau)^2} \frac{U^3}{h}, \tag{81}$$

using $C \approx 5.2$ and $\kappa \approx 0.41$. Again, there would be a log correction to the inertial scaling. A good engineer would adjust all the constants to fit the data as well as possible, but our purpose here was only to give a brief introduction to turbulence folklore.

References

[1] D. Acheson, *Elementary Fluid Dynamics*, Oxford University Press, 1990.

[2] F. H. Busse, *Bounds for turbulent shear flow*, Journal of Fluid Mechanics, 41 (1970), pp. 219–240.

[3] C. R. Doering and P. Constantin, *Energy dissipation in shear driven turbulence*, Phys. Rev. Lett., 69 (1992), pp. 1648–1651.

[4] J. Hamilton, J. Kim, and F. Waleffe, *Regeneration mechanisms of near-wall turbulence structures*, J. Fluid Mech., 287 (1995), pp. 317–348.

[5] L. N. Howard, *Heat transport by turbulent convection*, Journal of Fluid Mechanics, 17 (1963), pp. 405–432.

[6] L. N. Howard, *Bounds on flow quantities*, Annual Review of Fluid Mechanics, 4 (1972), pp. 473–494.

[7] G. Kawahara and S. Kida, *Periodic motion embedded in Plane Couette turbulence: regeneration cycle and burst*, J. Fluid Mech., 449 (2001), pp. 291–300.

[8] R. Kerswell, *Unification of variational principles for turbulent shear flows: the background method of doering-constantin and the mean-fluctuation formulation of howard-busse*, Physica D: Nonlinear Phenomena, 121 (1998), pp. 175 – 192.

[9] P. Kundu and I. Cohen, *Fluid Mechanics*, Academic Press, Elsevier, 2010.

[10] W. V. R. Malkus, *Outline of a theory of turbulent shear flow*, Journal of Fluid Mechanics, 1 (1956), pp. 521–539.

[11] T. Mullin, *Experimental studies of transition to turbulence in a pipe*, Annual Review of Fluid Mechanics, 43 (2011), pp. 1–24.

[12] S. C. Plasting and R. R. Kerswell, *Improved upper bound on the energy dissipation rate in plane Couette flow: the full solution to Busse's problem and the Constantin–Doering–Hopf problem with one-dimensional background field*, Journal of Fluid Mechanics, 477 (2003), pp. 363–379.

[13] S. Pope, *Turbulent Flows*, Cambridge University Press, 2000.

[14] P. Sagaut and C. Cambon, *Homogeneous Turbulence Dynamics*, Cambridge University Press, 2008.

[15] D. Viswanath, *Recurrent motions within plane Couette turbulence*, J. Fluid Mech., 580 (2007), pp. 339–358.