Boundary collapse in models of shear-flow transition

Norman R Lebovitz
Department of Mathematics
The University of Chicago
5734 S University Ave, Chicago IL 60637, USA

July 25, 2011

Abstract

We explore two low-dimensional dynamical systems modeling transition to turbulence in shear flows to try to understand the nature of the boundary ∂B of the basin of attraction B of the stable, laminar point at the origin of coordinates. Components of ∂B are found to exist in two types: one (the 'strong' type) separating B from a complementary set where orbits never relaminarize, and a second (the 'weak' type) separating B into two parts locally but not globally. For a boundary of weak type, orbits on each side relaminarize but may be distinguished from one another by features such as orbital complexity and time to relaminarize. The basin boundary may be of a single type, or may be a union of components of different types.

The models are parametrized and may transform from one type to another at a critical parameter value. In the models studied here the change from purely strong type to a union of the two types occurs via the collapse of two sheets of a strong boundary into a single sheet. This is accompanied, at the critical value of the parameter, by the appearance of a homoclinic orbit and the subsequent occurrence of a periodic orbit on the strong part of the boundary.

1 Introduction

Transition to turbulence in shear flows occurs as the relevant parameter, the Reynolds number R, increases beyond a critical value. This transition differs in important respects from the onset of instability in other familiar problems of hydrodynamics and indeed of other familiar problems in applied mathematics. An important difference is that the transition takes place while the unperturbed, laminar flow remains asymptotically stable ([7],[8]).

Another difference is that the transition may not be permanent, at least for a range of R values. In these cases an apparently complex motion occurs for a while but is then followed by relaminarization. This regime of a return to the laminar state after a complex motion has been found in numerical calculations and has been related to the existence of an 'edge' state ([1],[7]). Sometimes the complex motion has a chaotic character and one refers to the 'edge of chaos.' These features have been found both in the behavior of low-dimensional models of shear flows and in numerical treatments of the Navier-Stokes equations. The edge seems to be an invariant, codimension-one manifold in phase space separating relaminarizing orbits of two different types: orbits of one type relaminarize quickly, whereas those of the other type relaminarize more slowly and follow a more complicated trajectory than orbits of the first type (cf [10]). The edge state need not be chaotic (this is parameter-dependent) and we emphasize here its feature of dividing the basin of attraction into two parts.

The boundary ∂S of a given set S is defined as the set of points whose every neighborhood contains both points that are in S and points that are not in S. It is common to think of such a boundary as separating S from some other set. For example, on the real line the set $S = \{x : x > 0\}$ has the boundary consisting of the point $x = 0$ separating S from the complementary set where $x < 0$. On the other hand the set $S = \{x : |x| > 0\}$ has the same boundary $x = 0$, but this now separates S into two parts, each of which belongs to S. We'll refer below to the first kind of boundary as a *strong boundary*, effectively defining the limits of S, and the second kind as a *weak boundary* since it does not.

For shear flows the laminar flow is stable and therefore has a basin of attraction B, a connected, open set. An important key to transition lies in the nature of this set, or of its boundary ∂B. As we shall see in simple models below, ∂B may be of either of the kinds described above, or it may fail to be of a single kind but instead be the union of strong and weak sets. An 'edge' set has a natural explanation as a weak part of ∂B in that on either side of it orbits lie in B, i.e., relaminarize.

Finite-dimensional systems that mimic shear flows possess the following characteristics: they have only linear and quadratic terms, the linear terms feature a non-normal, stable matrix and the nonlinear terms conserve energy (cf [9]). We shall take the stable, laminar solution to be the origin of coordinates. Then an n-dimensional system of shearflow kind takes the form

$$\dot{x} = Ax + b(x),\ x \in R^n. \tag{1}$$

Here A is a stable, nonnormal matrix and the quadratic function $b(x)$ satisfies the condition $(x, b(x)) = 0$, where $(,)$ represents the usual scalar product in R^n.

In the present paper we present two examples of low-dimensional models of this kind in which an edge state appears. In these examples the edge state is formed from a strong basin boundary at a particular parameter value where two components of the strong boundary collapse onto one another forming a single weak boundary-component. This metamorphosis of phase space is accompanied by the appearance of a homoclinic orbit at the same, critical parameter value.

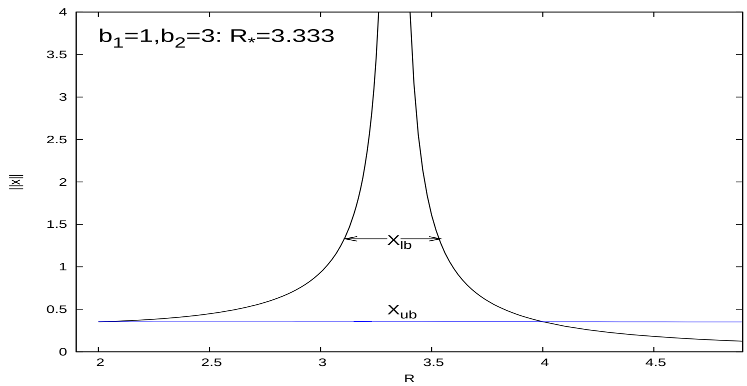

Figure 1: Norms of equilibrium points against R; $\|x\| = 0$ represents the stable, laminar point. The singular behavior as $R \rightarrow R_*$ ($R_* = 10/3$ in this example) can be inferred analytically when $(b_1, b_3) = (1, 3)$ in equation 2. Similar behavior is common to the cases when $b_1 b_2 > 0$.

2 A two-dimensional model

The following family of two-dimensional systems is of the shear-flow type of equation (1):

$$\dot{x}_1 = -\delta x_1 + x_2 + b_1 x_1 x_2 - b_2 x_2^2, \tag{2}$$

$$\dot{x}_2 = -\delta x_2 - b_1 x_1^2 + b_2 x_1 x_2. \tag{3}$$

Here δ is a sufficiently small parameter which we relate to the Reynolds number by the formula $\delta = 1/R$. The real parameters b_1, b_2 could be chosen arbitrarily, but in the numerical examples below we have chosen $b_1 = 1, b_2 = 3$. The laminar flow is represented by the equilibrium solution $(x_1, x_2) = (0, 0)$.

It is a simple matter to investigate the further equilibrium solutions of this system and their stability, and we shall state the results of this as needed, leaving details to the reader. We wish, however to point out one feature. We have chosen specific values for which the product $b_1 b_2 > 0$ and believe these choices to be robust, i.e., there would be no qualitative changes on altering b_1 and b_2 slightly. However, if we were to make the alternative choice $b_1 b_2 \leq 0$, the pattern of bifurcations found below would disappear. In older studies of two-dimensional systems ([3],[5]) this alternative choice was made.

The origin is the only equilibrium point if $R < 2$ and is globally, asymptotically stable in that case, but two further equilibrium points emerge via a saddle-node bifurcation at the value $R = R_{sn} = 2$. One of these, which we call X_{ub} (for 'upper branch'), is initially stable and the other, X_{lb} ('lower branch') unstable for all values of R: their norms are shown in Figure (1). The subscripts ub and lb stand for "upper branch" and "lower branch" respectively. A phase portrait for $R = 2.05$, just beyond the saddle-node value is shown in Figure (2). The basin of attraction of the origin, called B here and below, occupies most of phase space. Its boundary ∂B, which coincides with $SM(X_{lb})$, the stable manifold of X_{lb}, is simultaneously the boundary of the basin of attraction D of X_{ub}, and is therefore of the strong type: only on one side of it do orbits relaminarize (i.e., tend to the origin). Note that B becomes extremely thin for large values of x_1.

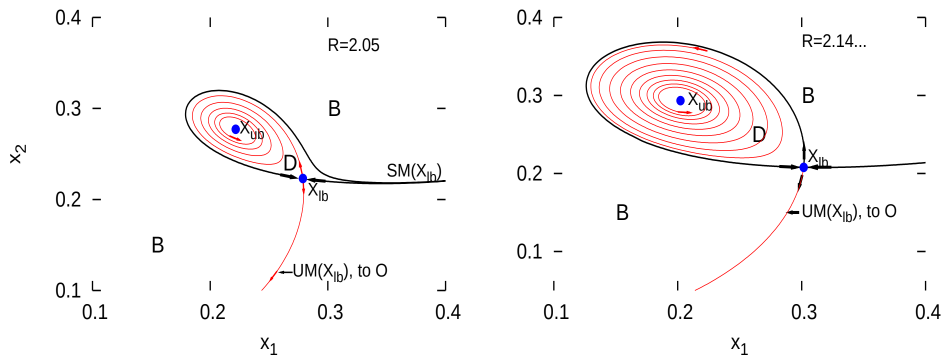

Figure 2: The values of the parameters are $b_1 = 1, b_2 = 3$. In the left-hand panel R is just greater than $R_{sn} = 2$. On the right, $R = R_h \approx 2.14$.

As R increases further the region D increases in size but a new critical value of $R = R_h \approx 2.14$ occurs where a homoclinic bifurcation takes place: the two branches of $SM(X_{lb})$ to the right of X_{lb} seen in Figure 2 coalesce into a single orbit, and D is now bounded by a homoclinic loop. This is also depicted in Figure 2. The part of the stable manifold of X_{lb} to the right of this point is now a weak part of the boundary, i.e., an edge: it separates one part of B from another. ∂B is now the union of strong- and weak-boundary-components; it continues to coincide with $SM(X_{lb})$.

Increasing R beyond R_h we find that the domain D is now bounded by a periodic orbit, indicated by P in Figure 3. Both arcs of $SM(X_{lb})$ now have the edge character. ∂B is the union of P (a strong boundary-component) with $SM(X_{lb})$ (an edge). The left-hand arc of the latter is both part of $SM(X_{lb})$ and the unstable manifold of the periodic orbit P; P is the α-limit set of the orbit consisting of the left-hand arc of $SM(X_{lb})$.

As R is increased D shrinks and the stability of X_{ub} weakens, i.e., the (negative) real parts of the eigenvalues get smaller in absolute value. This culminates in a further bifurcation point $R = R_{bh} = 2.5$[1] at which the real parts of the eigenvalues vanish and the domain D evanesces. For R just greater than R_{bh} essentially all of the phase space lies in B, i.e., essentially all orbits relaminarize. The exceptions are those lying on $SM(X_{lb})$ which again coincides with ∂B (see Figure 3). The basin boundary has now made a full transition from a strong to weak.

[1]This is a 'backward Hopf' bifurcation (thus R_{bh}), i.e., it would be a standard Hopf bifurcation if we changed t to $-t$ and ran R backwards.

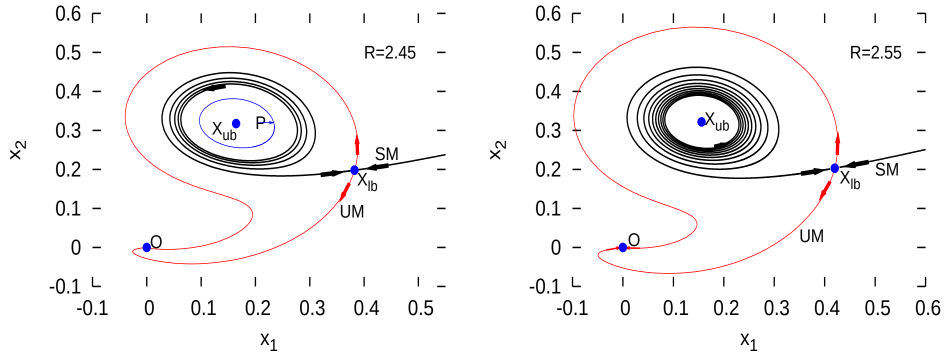

Figure 3: In the left-hand panel the value of R is just less than $R_{bh} = 2.5$, at which the stability of X_{ub} changes. The basin of attraction of X_{ub} has shrunk but persists and is now bounded by the periodic orbit P. On the right the value of R is slightly greater: X_{ub} is now unstable, the periodic orbit has disappeared via Hopf bifurcation, and the boundary of the basin of attraction of the origin is purely of the weak – or edge – type.

There is for the system (2,3) a final critical value for $R = R_\infty = 10/3$ at which a total alteration of phase space takes place. B is restored to finite size, and ∂B becomes a strong boundary (cf. Figure 4). In this case the points that lie outside B are carried by the flow to infinite distance as $t \to \infty$. This unphysical result emphasizes that conformity to the rules of model building (equation 1 above) is by no means sufficient for a realistic model. We pass on now to a better motivated choice.

3 A four-dimensional model

We consider in this section a four-dimensional model of a kind described by Waleffe ([12]) and previously studied from various standpoints (e.g.,[4],[6], [9]). We write it in a form in which the stable equilibrium point representing the laminar flow lies at the origin of coordinates The system of equations in question, which we'll refer to as W97, is the following:

$$\dot{x}_1 = -\delta r_1 x_1 + \sigma_1 x_4^2 - \sigma_2 x_2 x_3, \tag{4}$$

$$\dot{x}_2 = -\delta r_2 x_2 + \sigma_2 x_3 + \sigma_2 x_1 x_3 - \sigma_4 x_4^2, \tag{5}$$

$$\dot{x}_3 = -\delta r_3 x_3 + \sigma_3 x_4^2, \tag{6}$$

$$\dot{x}_4 = -\left(\sigma_1 + \delta r_4\right) x_4 + x_4 \left(\sigma_4 x_2 - \sigma_1 x_1 - \sigma_3 x_3\right). \tag{7}$$

Here $\delta = 1/R$ where R is the Reynolds number and the eight constants r_1 through σ_4 are all positive. This system conforms to the rules of model-building (see equation 1). It possesses the symmetry $S = \mathrm{diag}(1, 1, 1, -1)$ so a solution

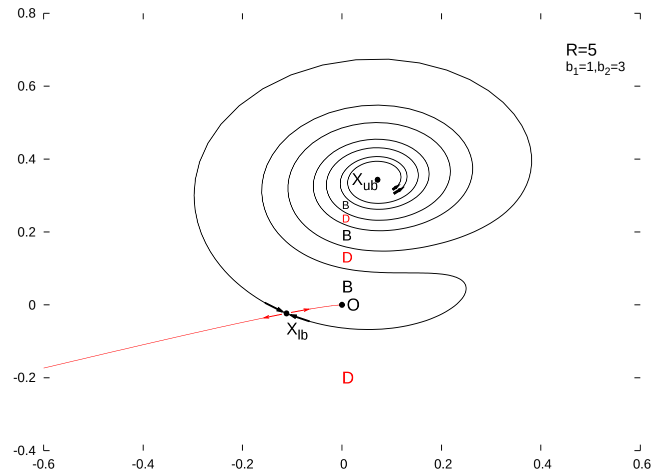

Figure 4: In the region D complementary to B all orbits tend to ∞.

$x(t)$ has a companion solution $\tilde{x}(t)$ obtained by reversing the sign of $x_4(t)$ and the plane $x_4 = 0$ is an invariant plane. For these reasons there is no loss of generality in considering only solutions for which $x_4(t) \geq 0$. It is not difficult to show that the invariant plane $x_4 = 0$ lies entirely in B, the basin of attraction of the origin. There are, for values of R greater than a saddle-node value R_{sn}, three equilibrium points: the origin O, X_{lb} and X_{ub}, the so-called lower-branch and upper-branch solutions. The origin is stable for all values of R, X_{lb} is unstable and has a single unstable direction for all $R > R_{sn}$ and the stability properties of X_{ub} depend on R.

There are canonical values for these constants that are often used. They are (cf. [4])

$$(\sigma_1, \sigma_2, \sigma_3, \sigma_4) = (0.31, 1.29, 0.22, 0.68)$$
$$(r_1, r_2, r_3, r_4) = (2.4649, 5.1984, 7.6729, 7.1289).$$

If these values are used, one finds (see [9]) that, for a moderate value $R = 190$, ∂B consists partly of a weak boundary component, the edge (containing the lower-branch equilibrium point X_{lb}), and partly of a strong boundary component (containing a periodic orbit). In the two-dimensional model considered in §2 this would correspond to values of the parameter greater than that at which the homoclinic bifurcation took place (R_h) but less than that of the Hopf bifurcation (R_{bh}). If, therefore, a similar homoclinic bifurcation takes place in the current, four-dimensional model, it must take place in a different parameter regime.

We choose different parameters for which ∂B is purely of the strong type, and then vary one of the parameters to see whether a phase-space transformation takes place. The reparametrization we have chosen is the result of a particular line of numerical experimentation, and there may well be other, perhaps less radical, reparametrizations, that would do equally well. We set all the positive constants in W97 equal to unity with the exception of σ_1, which we use as a bifurcation parameter. The Reynolds number R will be held fixed: $R = 15$. For small values of σ_1, we find numerically that ∂B is purely of the strong type, at least that part of ∂B containing X_{lb}. We note that, as in Figure (2) above, two sheets of ∂B become very near to each other in some parts of phase space, leaving only a narrow gap between them that belongs to the complement of B. This is seen in the left-hand panel of Figure (5), which shows a slice of the basin boundary passing through X_{lb} made by the hyperplane $x_1 = X_{lb1}, x_2 = X_{lb2}$. The parameter value, $\sigma_1 = 0.186$, is slightly less than the critical value.

As we consider successively larger values of σ_1 we see this gap closing and finally disappearing at a critical parameter value at which a loop homoclinic to X_{lb} appears. It is possible for this loop to form at the critical parameter value because the upper sheet of the basin boundary then just touches the lower sheet allowing stable and unstable eigenvectors of the system linearized around X_{lb} to lie simltaneously on the basin boundary. This is depicted in Figure (8) of [9], where slices of the basin boundary with the hyperplane $x_2 = X_{lb2}, x_3 = X_{lb3}$ are shown.

In Figure (5) the region marked Complement of B, is itself the basin of attraction of a stable periodic orbit (call it Q) lying outside B. For $\sigma_1 < \sigma_{1c}$, ∂B is the common boundary of the basin of attraction of the origin and of the basin of attraction of Q. For $\sigma_1 > \sigma_{1c}$ (right-hand panel of Figure 5), ∂B consists of two pieces: the edge, which contains X_{lb} and forms a weak component of ∂B, and the boundary of the basin of attraction of Q. The latter boundary now contains a further periodic orbit P which has come into existence via the homoclinic loop at $\sigma_1 = \sigma_{1c}$. Indeed, the boundary of the basin of attraction of Q appears to be $SM(P)$, the stable manifold of P, and the edge appears to be $SM(X_{lb})$, the stable manifold of the lower-branch equilibrium point.

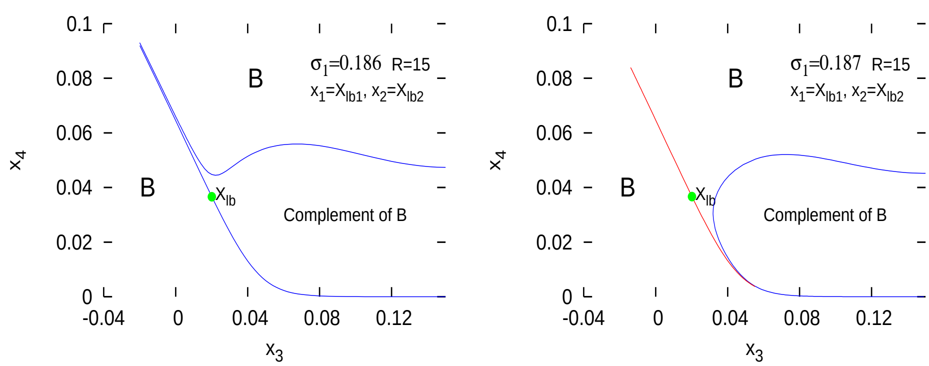

Figure 5: Closeup views of the part of phase space near X_{lb}, which is represented by a large dot. The left-hand diagram shows a slice of the basin boundary for $\sigma_1 < \sigma_{1c}$. The boundary is purely 'strong.' In the right-hand panel (post-critical) the sheet containing X_{lb} divides long-time relaminarizations from short-time relaminarizations.

4 Conclusions

Certain features of transition in shear flows can be captured by low-dimensional – even two-dimensional – models ([2],[3],[5],[11]). The present study adds to previously studied features a picture of an edge state. It confirms the view ([10]) that the edge is a codimension-one invariant set embedded in the basin of attraction of the laminar state. In the present paper the edge state is in fact the stable manifold of the unstable equilibrium point X_{lb}, or a subset of the latter. It emerges in the models studied here via the collapse of a strong-boundary component simultaneously with the emergence of a periodic orbit via a homoclinic bifurcation at a critical parameter value.

In the two-dimensional example studied in §2, this periodic orbit forms the boundary of the basin of attraction of a further stable equilibrium point X_{ub}, and the boundary ∂B of the basin of attraction of the origin consists of the union of this periodic orbit with the edge in the interval $R_h < R < R_{bh}$.

A similar result appears in a four-dimensional model studied in ([9]) and §3, wherein an edge state likewise makes an appearance via a homoclinic bifurcation. Whereas the equilibrium point X_{ub} is unstable for the parameter values considered, there is a stable periodic orbit Q with its own basin of attraction, and the boundary of the latter is the strong component of ∂B. In both this and the two-dimensional case the periodic orbit P that accompanies the appearance of the edge lies on that part of the basin boundary that remains of the strong type.

The evidence presented here is that the edge is formed in the collapse of two sheets into a single sheet. An alternative interpretation (wherein the pair of sheets becomes exquisitely close together – within computational precision – but remains separate even for $\sigma_1 > \sigma_{1c}$) was offered in ([9]) as being consistent with the numerical data. This alternative interpretation now seems implausible and the collapse of the two sheets into one seems far more likely. The evidence is particularly clear for the two-dimensional model where the two 'sheets' are curves.

Nevertheless the sudden reduction of an extensive, n-dimensional region of phase space to an (equally extensive) $(n-1)$-dimensional subspace (apparently, a manifold), seems quite remarkable. In higher-dimensional models, one might imagine a 'piecemeal' description wherein the two sheets touch, at least at first, (at $\sigma_1 = \sigma_{1c}$) along some curve. The nature of the diagrams is not favorable to such an hypothesis, however. In particular, in ([9]) the diagrams shown were slices of the basin boundary with x_2 and x_3 held fixed, whereas in Figure (5) we show a slice with x_1 and x_2 held fixed. One would not expect to see collapse simultaneously in both slices – as we do – if the 'piecemeal' explanation held.

In the two-dimensional model of §2 there is a final critical value R_∞ of the parameter beyond which the edge state has disappeared and the geometrical structure of phase space is consistent with the interpretation of a permanent, subcritical transition away from the laminar state. This of course may be a peculiarity of this model.

References

[1] M. Avila, A. Willis, B. Hof, On the transient nature of localized pipe flow turbulence, JFM 646 (2010) 127-136.

[2] J. Bagget, T. Driscoll, L. Trefethen, A mostly linear model of transition to turbulence, Phys. Fluids 7 (1995) 833-838.

[3] J. Bagget, L. Trefethen, Low-dimensional models of subcritical transition to turbulence, Phys. Fluids 9 (1997) 1043-1053.

[4] C. Cossu, An optimality condition on the minimum energy threshold in subcritical instabilities, C.R. Mecanique 333 (2005), 331-336.

[5] O. Dauchot,P. Manneville, Local versus global concepts in hydrodynamic stability theory, Jour Phys II France 7 (1997), 371-389.

[6] O. Dauchot, N. Vioujard, Phase space analysis of a dynamical model for the subcritical transition to turbulence in plane Couette flow, Eur. Phys. J. B 14 (2000), 377-381.

[7] B. Eckhardt, Turbulence transition in pipe flow: some open questions, Nonlinearity 21 (2008), T1-T11.

[8] S. Grossman, The onset of shear flow turbulence, Rev. Mod. Phys. 72 (2000), 603-618.

[9] N. Lebovitz, Shear-flow transition: the basin boundary, Nonlinearity 22 (2009), 2645-2655.

[10] J. Skufca, J. Yorke, B. Eckhardt, Edge of chaos in a parallel shear flow, Phys. Rev. Lett. 96 (2006), 174101.

[11] L. Trefethen, A. Trefethen, S. Reddy, S., T. Driscoll, Hydrodynamic stability without eigenvalues, Science 261 (1993), 578-584.

[12] F. Waleffe, On a self-sustaining process in shear flows, Phys. Fluids 9 (1997), 883-900.