Solving PDE's with FEniCS

Navier-Stokes equations

Chapter 20–21

Introduction to

Automated Modeling

with FEniCS

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The Navier-Stokes equations for the flow of a viscous, incompressible, Newtonian fluid can be written

$$\begin{aligned} -\Delta \mathbf{u} + \nabla p &= -R\left(\mathbf{u}\cdot\nabla\mathbf{u} + \mathbf{u}_t\right) \\ \nabla\cdot\mathbf{u} &= 0 \end{aligned} \tag{1}$$

in $\Omega \subset \mathbb{R}^d$, where $\mathbf{u}$ denotes the fluid velocity, p denotes the pressure, and R denotes the Reynolds number [2]. In our context, $R = 1/\eta$ where η denotes the fluid (kinematic) viscosity.

These equations must be supplemented by appropriate boundary conditions, such as the Dirichlet boundary conditions, $\mathbf{u} = \boldsymbol{\gamma}$ on $\partial\Omega$.

Navier-Stokes variational form

A complete variational formulation of (1) takes the form:
Find $\mathbf{u}$ such that $\mathbf{u} - \boldsymbol{\gamma} \in V$ and $p \in \Pi$ such that

$$
\begin{aligned}
a\left(\mathbf{u}, \mathbf{v}\right) + b\left(\mathbf{v}, p\right) & \\
+ R\Big(c\left(\mathbf{u}, \mathbf{u}, \mathbf{v}\right) + \left(\mathbf{u}_t, \mathbf{v}\right)_{L^2}\Big) &= 0 \quad \forall \mathbf{v} \in V\,, \\
b(\mathbf{u}, q) &= 0 \quad \forall q \in \Pi\,,
\end{aligned}
\tag{2}
$$

where e.g. $a(\cdot,\cdot)$ and $b(\cdot,\cdot)$ are the same as for Stokes:

$$
a(\mathbf{u}, \mathbf{v}) := \int_\Omega \sum_{i,j=1}^{n} u_{i,j} v_{i,j}\, d\mathbf{x}, \quad b(\mathbf{v}, q) := -\int_\Omega \sum_{i=1}^{n} v_{i,i} q\, d\mathbf{x},
$$

$(\cdot,\cdot)_{L^2}$ denotes the $L^2(\Omega)^d$-inner-product and new form is

$$
c(\mathbf{u}, \mathbf{v}, \mathbf{w}) := \int_\Omega (\mathbf{u} \cdot \nabla \mathbf{v}) \cdot \mathbf{w}\, d\mathbf{x}. \tag{3}
$$

Navier-Stokes nonlinear form

The spaces V and Π are the same as for the Stokes equations.

The trilinear form (3) has some special properties that reflect important physical properties of fluid dynamics.

To see these, need to derive some calculus identities.

For any vector-valued function $\mathbf{u}$ and scalar-valued function v, the product rule for derivatives gives

$$\nabla\cdot(\mathbf{u}\,v) = (\nabla\cdot\mathbf{u})v + \mathbf{u}\cdot\nabla v. \tag{4}$$

Applying product rule again gives

$$\nabla\cdot(\mathbf{u}\,v\,w) = (\nabla\cdot\mathbf{u})v\,w + (\mathbf{u}\cdot\nabla v)w + (\mathbf{u}\cdot\nabla w)v.$$

Properties of the nonlinear term

Repeat: $\quad \nabla\cdot(\mathbf{u}\,v\,w) = (\nabla\cdot\mathbf{u})v\,w + (\mathbf{u}\cdot\nabla v)w + (\mathbf{u}\cdot\nabla w)v.$

Thus if we apply the divergence theorem we get

$$\oint_{\partial\Omega}(\mathbf{u}\cdot\mathbf{n})v\,w\,ds = \int_\Omega \nabla\cdot(\mathbf{u}\,v\,w)\,d\mathbf{x}$$
$$= \int_\Omega(\nabla\cdot\mathbf{u})v\,w\,d\mathbf{x} + \int_\Omega(\mathbf{u}\cdot\nabla v)w\,d\mathbf{x} + \int_\Omega(\mathbf{u}\cdot\nabla w)v\,d\mathbf{x}.$$

Thus if $\mathbf{u}$ satisfies the divergence constraint $\nabla\cdot\mathbf{u} = 0$ in (1) and the product $(\mathbf{u}\cdot\mathbf{n})\,v\,w$ vanishes on $\partial\Omega$, we have

$$\int_\Omega(\mathbf{u}\cdot\nabla v)w\,d\mathbf{x} = -\int_\Omega(\mathbf{u}\cdot\nabla w)v\,d\mathbf{x}. \tag{5}$$

Properties of the nonlinear term

Suppose now that $\mathbf{v}$ and $\mathbf{w}$ are vector-valued functions, and $\mathbf{u} \cdot \mathbf{n}$ or $\mathbf{v}$ or $\mathbf{w}$ vanishes at each point on $\partial\Omega$.

Applying (5), we find

$$\int_\Omega (\mathbf{u} \cdot \nabla \mathbf{v}) \cdot \mathbf{w} \, d\mathbf{x} = - \int_\Omega (\mathbf{u} \cdot \nabla \mathbf{w}) \cdot \mathbf{v} \, d\mathbf{x}. \tag{6}$$

Then the trilinear form (3) is antisymmetric in the last two arguments:

$$c(\mathbf{u}, \mathbf{v}, \mathbf{w}) = -c(\mathbf{u}, \mathbf{w}, \mathbf{v}). \tag{7}$$

In particular, if $\nabla \cdot \mathbf{u} = 0$ and $\mathbf{v}$ vanishes on $\partial\Omega$, then

$$c(\mathbf{u}, \mathbf{v}, \mathbf{v}) = 0.$$

Navier-Stokes time-stepping

Simple time-stepping scheme for Navier-Stokes equations (1) is implicit with respect to linear terms and explicit with respect to nonlinear terms.

However, fully implicit schemes have better stability properties.

The workhorse schmes for solving the time-dependent Navier-Stokes equations are implicit.

The simplest of these is the implicit Euler scheme, which is the lowest-order backwards differentiation (BDF) scheme.

The implicit Euler time-stepping scheme for the Navier-Stokes equations can be defined as follows.

Expressed in variational form, it is

$$
\begin{aligned}
a\left(\mathbf{u}^{\ell}, \mathbf{v}\right) + b\left(\mathbf{v}, p^{\ell}\right) + R\, c\left(\mathbf{u}^{\ell}, \mathbf{u}^{\ell}, \mathbf{v}\right) & \\
+ \frac{R}{\Delta t}\left(\mathbf{u}^{\ell} - \mathbf{u}^{\ell-1}, \mathbf{v}\right)_{L^2} &= 0, \\
b\left(\mathbf{u}^{\ell}, q\right) &= 0,
\end{aligned} \tag{8}
$$

where $\mathbf{v}$ varies over all V (or V_h) and q varies over all Π (or Π_h) and Δt denotes the time-step size.

More efficient time-stepping schemes will take a similar form, such as the backwards differentiation schemes. In particular, (8) is the first-order backwards differentiation scheme.

At each time step, one has a problem to solve for $(\mathbf{u}^\ell, p^\ell)$ with the form $\tilde{a}(\cdot,\cdot)$ is

$$\tilde{a}(\mathbf{u},\mathbf{v}) := a(\mathbf{u},\mathbf{v}) + \tau\left(\mathbf{u},\mathbf{v}\right)_{L^2}, \tag{9}$$

where the constant $\tau = R/\Delta t$.

It takes the form

$$\begin{aligned} \tilde{a}\left(\mathbf{u}^\ell,\mathbf{v}\right) + b\left(\mathbf{v},p^\ell\right) + R\,c\left(\mathbf{u}^\ell,\mathbf{u}^\ell,\mathbf{v}\right) &= \tau\left(\mathbf{u}^{\ell-1},\mathbf{v}\right)_{L^2}, \\ b\left(\mathbf{u}^\ell,q\right) &= 0. \end{aligned} \tag{10}$$

However, (10) is nonlinear, so an algorithm must be chosen to linearize it.

Fixed-point iteration

One of the simplest solution methods is the fixed-point iteration, which takes the form:

$$\begin{aligned} \tilde{a}\left(\mathbf{u}^{\ell,k}, \mathbf{v}\right) + b\left(\mathbf{v}, p^{\ell,k}\right) &= -R\, c\left(\mathbf{u}^{\ell,k-1}, \mathbf{u}^{\ell,k-1}, \mathbf{v}\right) \\ &\quad + \tau\left(\mathbf{u}^{\ell-1}, \mathbf{v}\right)_{L^2}, \\ b\left(\mathbf{u}^{\ell,k}, q\right) &= 0. \end{aligned} \tag{11}$$

This iteration can be started with an extrapolated value, e.g., $\mathbf{u}^{\ell,0} := 2\mathbf{u}^{\ell-1} - \mathbf{u}^{\ell-2}$, and once convergence is achieved, we set $\mathbf{u}^{\ell} = \mathbf{u}^{\ell,k}$.

Note that we have $\mathbf{u}^{\ell,k} = \boldsymbol{\gamma}$ on $\partial\Omega$, that is, $\mathbf{u}^{\ell,k} = \mathbf{u}_0^{\ell,k} + \boldsymbol{\gamma}$ where $\mathbf{u}_0^{\ell,k} \in V$.

Fixed-point iteration

The variational problem for $\mathbf{u}^{\ell,k}$ can be written:
Find $\mathbf{u}_0^{\ell,k} \in V$ and $p \in \Pi$ such that

$$\begin{aligned} \tilde{a}\big(\mathbf{u}_0^{\ell,k}, \mathbf{v}\big) + b\big(\mathbf{v}, p\big) = -\tilde{a}\big(\boldsymbol{\gamma}, \mathbf{v}\big) \\ -R\, c\big(\mathbf{u}^{\ell,k-1}, \mathbf{u}^{\ell,k-1}, \mathbf{v}\big) + \tau\big(\mathbf{u}^{\ell}, \mathbf{v}\big)_{L^2} \quad \forall \mathbf{v} \in V \\ b\big(\mathbf{u}_0^{\ell,k}, q\big) = -\, b\big(\boldsymbol{\gamma}, q\big) \quad \forall q \in \Pi. \end{aligned} \tag{12}$$

This is of the form

$$\begin{aligned} F(\mathbf{v}) = -\tilde{a}\big(\boldsymbol{\gamma}, \mathbf{v}\big) - R\, c\big(\mathbf{u}^{\ell,k-1}, \mathbf{u}^{\ell,k-1}, \mathbf{v}\big) \\ + \tau\big(\mathbf{u}^{\ell-1}, \mathbf{v}\big)_{L^2} \quad \forall \mathbf{v} \in V \\ G\big(q\big) = -\, b\big(\boldsymbol{\gamma}, q\big) \quad \forall q \in \Pi. \end{aligned} \tag{13}$$

Boundary data $\boldsymbol{\gamma}$ appears in both F and G.

The nonlinear time-stepping scheme (10) has excellent stability properties.

To see why, let us assume for simplicity that the boundary data γ is zero. Then

$$\tilde{a}\big(\mathbf{u}^\ell, \mathbf{u}^\ell\big) = -R\,c\big(\mathbf{u}^\ell, \mathbf{u}^\ell, \mathbf{u}^\ell\big) + \tau\big(\mathbf{u}^{\ell-1}, \mathbf{u}^\ell\big)_{L^2} = \tau\big(\mathbf{u}^{\ell-1}, \mathbf{u}^\ell\big)_{L^2}.$$

Then the Cauchy-Schwarz inequality implies

$$\begin{aligned} \tilde{a}\big(\mathbf{u}^\ell, \mathbf{u}^\ell\big) &\le \tau \|\mathbf{u}^{\ell-1}\|_{L^2(\Omega)} \|\mathbf{u}^\ell\|_{L^2(\Omega)} \\ &\le \tfrac{1}{2}\tau\Big(\|\mathbf{u}^{\ell-1}\|^2_{L^2(\Omega)} + \|\mathbf{u}^\ell\|^2_{L^2(\Omega)}\Big) \end{aligned}$$

Stability of the exact solution

Subtracting $\frac{1}{2}\tau\|\mathbf{u}^\ell\|^2_{L^2(\Omega)}$ from both sides yields

$$
\begin{aligned}
a\big(\mathbf{u}^\ell, \mathbf{u}^\ell\big) + \tfrac{1}{2}\tau\|\mathbf{u}^\ell\|^2_{L^2(\Omega)} &\le \tfrac{1}{2}\tau\|\mathbf{u}^{\ell-1}\|^2_{L^2(\Omega)} \\
&\le a\big(\mathbf{u}^{\ell-1}, \mathbf{u}^{\ell-1}\big) + \tfrac{1}{2}\tau\|\mathbf{u}^{\ell-1}\|^2_{L^2(\Omega)}.
\end{aligned}
\tag{14}
$$

Thus $\mathbf{u}^\ell$ is non-increasing in the norm

$$
\|\mathbf{v}\|_\tau = \sqrt{a(\mathbf{v}, \mathbf{v}) + \tfrac{1}{2}\tau\|\mathbf{v}\|^2_{L^2(\Omega)}}.
$$

Therefore

$$
\|\mathbf{u}^\ell\|_\tau \le \|\mathbf{u}^\ell\|_\tau,
$$

that is, subsequent time steps are bounded by the initial value in this norm.

Convergence of fixed-point iteration

The convergence of the iterative scheme (11) can be analyzed as follows.

Subtracting two consecutive versions of (11), we find the following formula for $\mathbf{e}^k := \mathbf{u}^{\ell,k} - \mathbf{u}^{\ell,k-1}$:

$$\begin{aligned}
\tilde{a}\left(\mathbf{e}^k, \mathbf{e}^k\right) &= R\Big(c\left(\mathbf{u}^{\ell,k-1}, \mathbf{u}^{\ell,k-1}, \mathbf{e}^k\right) - c\left(\mathbf{u}^{\ell,k-2}, \mathbf{u}^{\ell,k-2}, \mathbf{e}^k\right)\Big) \\
&= R\Big(c\left(\mathbf{u}^{\ell,k-1}, \mathbf{u}^{\ell,k-1}, \mathbf{e}^k\right) - c\left(\mathbf{u}^{\ell,k-2}, \mathbf{u}^{\ell,k-1}, \mathbf{e}^k\right) + \\
&\qquad c\left(\mathbf{u}^{\ell,k-2}, \mathbf{u}^{\ell,k-1}, \mathbf{e}^k\right) - c\left(\mathbf{u}^{\ell,k-2}, \mathbf{u}^{\ell,k-2}, \mathbf{e}^k\right)\Big) \\
&= R\Big(c\left(\mathbf{e}^{k-1}, \mathbf{u}^{\ell,k-1}, \mathbf{e}^k\right) + c\left(\mathbf{u}^{\ell,k-2}, \mathbf{e}^{k-1}, \mathbf{e}^k\right)\Big).
\end{aligned}$$

Convergence of fixed-point iteration

So we have

$$\tilde{a}\left(\mathbf{e}^k, \mathbf{e}^k\right) = R\Big(c\left(\mathbf{e}^{k-1}, \mathbf{u}^{\ell,k-1}, \mathbf{e}^k\right) + c\left(\mathbf{u}^{\ell,k-2}, \mathbf{e}^{k-1}, \mathbf{e}^k\right)\Big).$$

From the Cauchy-Schwarz inequality, we find

$$|c(\mathbf{u}^{\ell,k-2}, \mathbf{e}^{k-1}, \mathbf{e}^k)| \leq \|\mathbf{u}^{\ell,k-2}\|_{L^\infty(\Omega)} a(\mathbf{e}^{k-1}, \mathbf{e}^{k-1})^{1/2} (\mathbf{e}^k, \mathbf{e}^k)^{1/2}.$$

From (7) and the Cauchy-Schwarz inequality, we find

$$\begin{aligned} |c(\mathbf{e}^{k-1}, \mathbf{u}^{\ell,k-1}, \mathbf{e}^k)| &= |c(\mathbf{e}^{k-1}, \mathbf{e}^k, \mathbf{u}^{\ell,k-1})| \\ &\leq \|\mathbf{u}^{\ell,k-1}\|_{L^\infty(\Omega)} (\mathbf{e}^{k-1}, \mathbf{e}^{k-1})^{1/2} (a(\mathbf{e}^k, \mathbf{e}^k))^{1/2}. \end{aligned}$$

Convergence of fixed-point iteration

Estimating the terms with the "c" form in them, we find

$$
\begin{aligned}
\tilde{a}\left(\mathbf{e}^k,\mathbf{e}^k\right) &\le R\|\mathbf{u}^{\ell,k-1}\|_{L^\infty(\Omega)}\left(\mathbf{e}^{k-1},\mathbf{e}^{k-1}\right)^{1/2}\left(a\left(\mathbf{e}^k,\mathbf{e}^k\right)\right)^{1/2} \\
&+ R\|\mathbf{u}^{\ell,k-2}\|_{L^\infty(\Omega)}\left(a\left(\mathbf{e}^{k-1},\mathbf{e}^{k-1}\right)\right)^{1/2}\left(\mathbf{e}^k,\mathbf{e}^k\right)^{1/2} \\
&\le \tfrac{1}{2}\left(R\|\mathbf{u}^{\ell,k-1}\|_{L^\infty(\Omega)}\right)^2\left(\mathbf{e}^{k-1},\mathbf{e}^{k-1}\right) + \tfrac{1}{2}a\left(\mathbf{e}^k,\mathbf{e}^k\right) \\
&+ \frac{1}{2\tau}\left(R\|\mathbf{u}^{\ell,k-2}\|_{L^\infty(\Omega)}\right)^2 a\left(\mathbf{e}^{k-1},\mathbf{e}^{k-1}\right) + \frac{\tau}{2}\left(\mathbf{e}^k,\mathbf{e}^k\right) \\
&= \tfrac{1}{2}\left(R\|\mathbf{u}^{\ell,k-1}\|_{L^\infty(\Omega)}\right)^2\left(\mathbf{e}^{k-1},\mathbf{e}^{k-1}\right) \\
&+ \frac{1}{2\tau}\left(R\|\mathbf{u}^{\ell,k-2}\|_{L^\infty(\Omega)}\right)^2 a\left(\mathbf{e}^{k-1},\mathbf{e}^{k-1}\right) + \tfrac{1}{2}\tilde{a}\left(\mathbf{e}^k,\mathbf{e}^k\right)
\end{aligned}
$$

Convergence of fixed-point iteration

So

$$\tilde{a}\left(\mathbf{e}^k, \mathbf{e}^k\right) \le \tfrac{1}{2}\left(R\|\mathbf{u}^{\ell,k-1}\|_{L^\infty(\Omega)}\right)^2 \left(\mathbf{e}^{k-1}, \mathbf{e}^{k-1}\right)$$

$$+ \frac{1}{2\tau}\left(R\|\mathbf{u}^{\ell,k-2}\|_{L^\infty(\Omega)}\right)^2 a\left(\mathbf{e}^{k-1}, \mathbf{e}^{k-1}\right) + \tfrac{1}{2}\tilde{a}\left(\mathbf{e}^k, \mathbf{e}^k\right)$$

$$\le \frac{R^2}{2\tau}\left(\sum_{j=1}^{2}\|\mathbf{u}^{\ell,k-j}\|^2_{L^\infty(\Omega)}\right)\tilde{a}\left(\mathbf{e}^{k-1}, \mathbf{e}^{k-1}\right) + \tfrac{1}{2}\tilde{a}\left(\mathbf{e}^k, \mathbf{e}^k\right).$$

Therefore (note $R^2/\tau = R\Delta t$)

$$\tilde{a}\left(\mathbf{e}^k, \mathbf{e}^k\right) \le \frac{R^2}{\tau}\left(\sum_{j=1}^{2}\|\mathbf{u}^{\ell,k-j}\|^2_{L^\infty(\Omega)}\right)\tilde{a}\left(\mathbf{e}^{k-1}, \mathbf{e}^{k-1}\right).$$

So fixed-point iteration converges provided Δt is small enough and $\|\mathbf{u}^{\ell,j}\|_{L^\infty(\Omega)}$ stay bounded.

Compatibility conditions govern solution smoothness.

For the Navier-Stokes equations, there are local compatibility conditions like those found for the heat equation.

However, there are also non-local compatibility conditions for the Navier-Stokes equations, and we describe them subsequently.

Local Compatibility Conditions

There are local *compatibility conditions* for boundary and initial data for Navier-Stokes equations similar to ones for the heat equation to have a smooth solution.

These can be derived again from the observation that that the values of u on the spatial boundary have been specified twice at $t = 0$.

The first condition is simply

$$\mathbf{u_0}(\mathbf{x}) = \boldsymbol{\gamma}(\mathbf{x}) \quad \forall \mathbf{x} \in \partial\Omega. \tag{15}$$

Additional conditions arrise by using the differential equation at $t = 0$ and for $x \in \partial\Omega$, but we post-pone temporarily deriving one of these.

Incompressibility Condition

However, we find a new type of condition, namely,

$$\nabla \cdot \mathbf{u_0} = 0. \tag{16}$$

Although this condition is quite obvious, it does pose a significant constraint on the initial data.

If either compatibility is not satisfied wild oscillations will result near $t = 0$ and $x \in \partial\Omega$.

In such a nonlinear problem, this can cause completely wrong results to occur.

Compatibility conditions (15) and (16) do not have to be satisfied for the Navier-Stokes equations (1) to be well posed in the weak sense.

Another condition arises due to the incompressibility (or divergence-free) condition:

$$\oint_{\partial\Omega} \mathbf{n} \cdot \boldsymbol{\gamma} = 0. \tag{17}$$

This simply means that the amount of fluid coming into the domain must balance the amount of fluid going out of the domain: the net mass flux into the domain is zero.

If this compatibility condition is violated, then the solution can clearly not have divergence zero.

Local Compatibility Conditions

There is a unique solution in any case, but the physical model may be incorrect as a result if it is supposed to have a smooth solution.

Compatibility conditions are a very subtle form of constraint on model quality.

The compatibility conditions (15) and (16) are described in terms of local differential-algebraic constraints.

However, we will see that there are other compatibility conditions that lead to global constraints that may be extremely hard to verify or satisfy in practice.

A nonlocal compatibility condition

Simply applying the first equation in (1) on $\partial\Omega$ at $t = 0$ we find

$$-\Delta \mathbf{u_0} + \nabla p_0 = -R\left(\boldsymbol{\gamma} \cdot \nabla \mathbf{u_0} + \boldsymbol{\gamma}'\right) \text{ on } \partial\Omega \tag{18}$$

where p_0 denotes the initial pressure.

Since p_0 is not part of the data, it might be reasonable to assume that the pressure initially would just adjust to insure smoothness of the system, i.e., satisfaction of (18), which we can re-write as

$$\nabla p_0 = \Delta \mathbf{u_0} - R\left(\boldsymbol{\gamma} \cdot \nabla \mathbf{u_0} + \boldsymbol{\gamma}'\right) \text{ on } \partial\Omega \tag{19}$$

Nonlocal Compatibility Conditions

However, taking the divergence of the first equation in (1) at $t = 0$ we find

$$\Delta p_0 = -R\nabla\cdot\left(\mathbf{u_0}\cdot\nabla\mathbf{u_0}\right) \text{ in } \Omega. \tag{20}$$

Thus p_0 must satisfy a Laplace equation with the full gradient specified on the bounary by (19).

Note that, if $\nabla\cdot\mathbf{v} = \mathbf{0}$, then

$$\nabla\cdot\left(\mathbf{v}\cdot\nabla\mathbf{v}\right) = \sum_{i,j} v_{i,j}v_{j,i} = \nabla\mathbf{v} : \nabla\mathbf{v}^t \text{ in } \Omega. \tag{21}$$

Thus we can write (20) as

$$\Delta p_0 = -R\left(\nabla\mathbf{u_0} : \nabla\mathbf{u_0}^t\right) \text{ in } \Omega.$$

Nonlocal Compatibility Conditions

Summarizing, we have

$$\Delta p_0 = -R\left(\nabla \mathbf{u_0} : \nabla \mathbf{u_0}^t\right) \text{ in } \Omega.$$

together with the boundary conditions

$$\nabla p_0 = \Delta \mathbf{u_0} - R\left(\boldsymbol{\gamma} \cdot \nabla \mathbf{u_0} + \boldsymbol{\gamma}'\right) \text{ on } \partial\Omega.$$

This is an over-specified system (one too many boundary conditions), so not all $\mathbf{u_0}$ will satisfy it.

The only simple way to satisfy both (19) and (20) is to have $\mathbf{u_0} \equiv \mathbf{0}$, that is to start with the fluid at rest.

Nonlocal Compatibility Conditions

For $R > 0$, it is easy to see that the system given by (19) and (20) is over-specified since γ' can be chosen arbitrarily.

A simple way to avoid the nonlocal compatibility condition is to start the simulation with

- initial data $\mathbf{u}_0 = \mathbf{0}$ and
- boundary data $\boldsymbol{\gamma}(0) = \boldsymbol{\gamma}'(0) = \mathbf{0}$.

Moreover, such an approach may be more appropriate physically.

Consider the time-independent Navier-Stokes equations in variational form:

$$F_{\mathbf{v}}(\mathbf{u}) = a(\mathbf{u}, \mathbf{v}) + Rc(\mathbf{u}, \mathbf{u}, \mathbf{v}) = 0 \quad \forall \mathbf{v} \in Z, \tag{22}$$

together with the incompressibility constraint $\mathbf{u} \in Z$, that is, equivalently $\nabla \cdot \mathbf{u} = 0$ or $b(\mathbf{u}, q) = 0$ for all $q \in \Pi$.

In addition, we assume that we have Dirichlet boundary conditions $\mathbf{u} = \mathbf{g}$ on $\partial\Omega$.

This is a nonlinear system of equations posed on the affine subspace $Z + \mathbf{g}$.

Variety of nonlinear solvers that can be applied.

But it is useful to see what Newton's method looks like for this system.

We compute the Jacobian of F by taking limits of difference quotients:

$$\begin{aligned} F_{\mathbf{v}}(\mathbf{u}+\epsilon\mathbf{w}) - F_{\mathbf{v}}(\mathbf{u}) = \epsilon\, a(\mathbf{w},\mathbf{v}) \\ + R\big(c(\mathbf{u}+\epsilon\mathbf{w},\mathbf{u}+\epsilon\mathbf{w},\mathbf{v}) - c(\mathbf{u},\mathbf{u},\mathbf{v})\big). \end{aligned}$$

We are differentiating in the affine space $Z+\mathbf{g}$, so perturbation $\mathbf{w}$ satisfies $\mathbf{w}=\mathbf{0}$ on $\partial\Omega$, as well as $\nabla\cdot\mathbf{w}=0$ (in short, $\mathbf{w}\in Z$).

Expanding

$$
\begin{aligned}
c(\mathbf{u}+\epsilon\mathbf{w},\mathbf{u}+\epsilon\mathbf{w},\mathbf{v}) &= c(\mathbf{u},\mathbf{u}+\epsilon\mathbf{w},\mathbf{v})+\epsilon\, c(\mathbf{w},\mathbf{u}+\epsilon\mathbf{w},\mathbf{v}) \\
&= c(\mathbf{u},\mathbf{u},\mathbf{v})+\epsilon\, c(\mathbf{u},\mathbf{w},\mathbf{v})+\epsilon\big(c(\mathbf{w},\mathbf{u},\mathbf{v})+\epsilon\, c(\mathbf{w},\mathbf{w},\mathbf{v})\big) \\
&= c(\mathbf{u},\mathbf{u},\mathbf{v})+\epsilon\big(c(\mathbf{u},\mathbf{w},\mathbf{v})+c(\mathbf{w},\mathbf{u},\mathbf{v})\big)+\epsilon^2 c(\mathbf{w},\mathbf{w},\mathbf{v}).
\end{aligned}
$$

Thus

$$
\begin{aligned}
F_{\mathbf{v}}(\mathbf{u}+\epsilon\mathbf{w})-F_{\mathbf{v}}(\mathbf{u}) = \epsilon\big(a(\mathbf{w},\mathbf{v}) \\
+R(c(\mathbf{u},\mathbf{w},\mathbf{v})+c(\mathbf{w},\mathbf{u},\mathbf{v}))\big)+R\epsilon^2 c(\mathbf{w},\mathbf{w},\mathbf{v}).
\end{aligned}
$$

Define a new variational form

$$
\hat{c}(\mathbf{u},\mathbf{w},\mathbf{v}) = c(\mathbf{u},\mathbf{w},\mathbf{v})+c(\mathbf{w},\mathbf{u},\mathbf{v}). \tag{23}
$$

Navier-Stokes variational form

Therefore

$$
\begin{aligned}
J_F(\mathbf{u})_{\mathbf{v},\mathbf{w}} &= \lim_{\epsilon\to 0} \epsilon^{-1}\big(F(\mathbf{u}+\epsilon\mathbf{w},\mathbf{v}) - F(\mathbf{u},\mathbf{v})\big) \\
&= a(\mathbf{w},\mathbf{v}) + R\,\hat{c}(\mathbf{u},\mathbf{w},\mathbf{v}).
\end{aligned} \tag{24}
$$

Thus Newton's method takes the form
Find $\mathbf{w} \in Z$ such that

$$
\begin{gathered}
a(\mathbf{w},\mathbf{v}) + R\,\hat{c}(\mathbf{u}^n,\mathbf{w},\mathbf{v}) = F(\mathbf{u}^n,\mathbf{v}) \quad \forall \mathbf{v} \in Z, \\
\text{Set} \quad \mathbf{u}^{n+1} = \mathbf{u}^n - \mathbf{w}.
\end{gathered} \tag{25}
$$

Initial $\mathbf{u}^0 \in Z + \mathbf{g}$ could come from Stokes equations satisfying the boundary conditions, $\mathbf{u}^0 \in Z + \mathbf{g}$.

Note that $\hat{c}(\mathbf{u}, \mathbf{w}, \mathbf{v}) = \hat{c}(\mathbf{w}, \mathbf{u}, \mathbf{v})$, and we find

$$\begin{aligned} \hat{c}(\mathbf{u}, \mathbf{w}, \mathbf{w}) &= c(\mathbf{u}, \mathbf{w}, \mathbf{w}) + c(\mathbf{w}, \mathbf{u}, \mathbf{w}) \\ &= c(\mathbf{w}, \mathbf{u}, \mathbf{w}) = \int_\Omega \mathbf{w}^t (\nabla \mathbf{u}) \mathbf{w} \, d\mathbf{x}. \end{aligned} \tag{26}$$

For larger R, use continuation method increasing R successively.

Take $\mathbf{u}^0$ at each step to be solution for previous R.

System (25) can be solved by iterated penalty method.

Navier-Stokes flows

We give here some important examples of Navier-Stokes flow.

The first ones are exact, but they are also the basis for many important applications.

We begin with two exact solutions.

Then we consider ones where no analytical form is available.

Poiseuille flow

We begin with flow in a channel.

Surprisingly, solution is same for all Reynolds numbers.

Recall the notion of a two-dimensional channel Ω:

$$\Omega = \left\{(x, y) \in \mathbb{R}^2 \ : \ 0 \leq x \leq L, \ 0 \leq y \leq 1\right\}.$$

Assumed that flow is negligible in the third dimension across the width of the canal.

Also recall Poiseuille flow, defined by

$$\mathbf{u}(x, y) = (u(x, y), v(x, y)) = (\tfrac{1}{2}y(1 - y), 0),$$

for $(x, y) \in \Omega$.

Navier-Stokes exact solution

Remarkably, this is also an exact solution of the Navier-Stokes equations, independent of the Reynolds number.

This is because $v \equiv 0$ and the nonlinear term vanishes:

$$\mathbf{u} \cdot \nabla \mathbf{u} = u(\mathbf{u}_x) = \mathbf{0},$$

since $\mathbf{u}$ is independent of x.

This $\mathbf{u}$ satisfies homogeneous Dirichlet boundary conditions on both the top and bottom of the channel.

As before, the pressure is given by $p(x, y) = x$.

Solution verification

Then

$$-\Delta \mathbf{u}(x,y) + \mathbf{u}\cdot\nabla\mathbf{u} + \nabla p(x,y) = -\Delta\mathbf{u}(x,y) + \nabla p(x,y) = 0$$

for $(x,y) \in \Omega$.

Three-dimensional flow in a pipe (cylinder) is similar and also is named for Poiseuille.

We can think of these flows as being driven by the non-zero pressure gradient.

Unfortunately, the Stokes cross does not generalize to the Navier-Stokes equations.

Exact solution of the Navier-Stokes equations in a converging or diverging duct.

The flow is defined by

$$\mathbf{u}(x, y) := \nu \, \frac{u(\mathrm{atan}(y/x))}{x^2 + y^2} \, \mathbf{x} \, , \quad \mathbf{x} = (x, y) \in \Omega$$

where u solves the ODE

$$-\frac{d^2 u}{d\xi^2}(\xi) + 4\, u(\xi) + 6\, u(\xi)^2 = f(\xi) \quad \text{for } \xi \in [0, \theta], \qquad u(0) = 0, \;\; u'(\theta) = 0. \tag{27}$$

With $f = C$. Reynolds number related to C.

Driven cavity for Navier-Stokes

Let $\Omega = [0, 1]^2$ and let Γ denote the top of the square:

$$\Gamma = \{(x, 1) \ : \ 0 \leq x \leq 1\}.$$

Define Dirichlet boundary conditions by

$$\mathbf{u}(x, 1) = 1 \text{ on } \Gamma, \quad \mathbf{u} = \mathbf{0} \text{ on } \partial\Omega\backslash\Gamma.$$

Resulting Navier-Stokes problem with this boundary condition is driven cavity problem for Navier-Stokes.

Denote the solution by $\mathbf{u}_R$ where R is Rynolds number.

The solution of the Stokes driven cavity problem is the same as $\mathbf{u}_0$.

Driven cavity for Navier-Stokes: $R = 50$

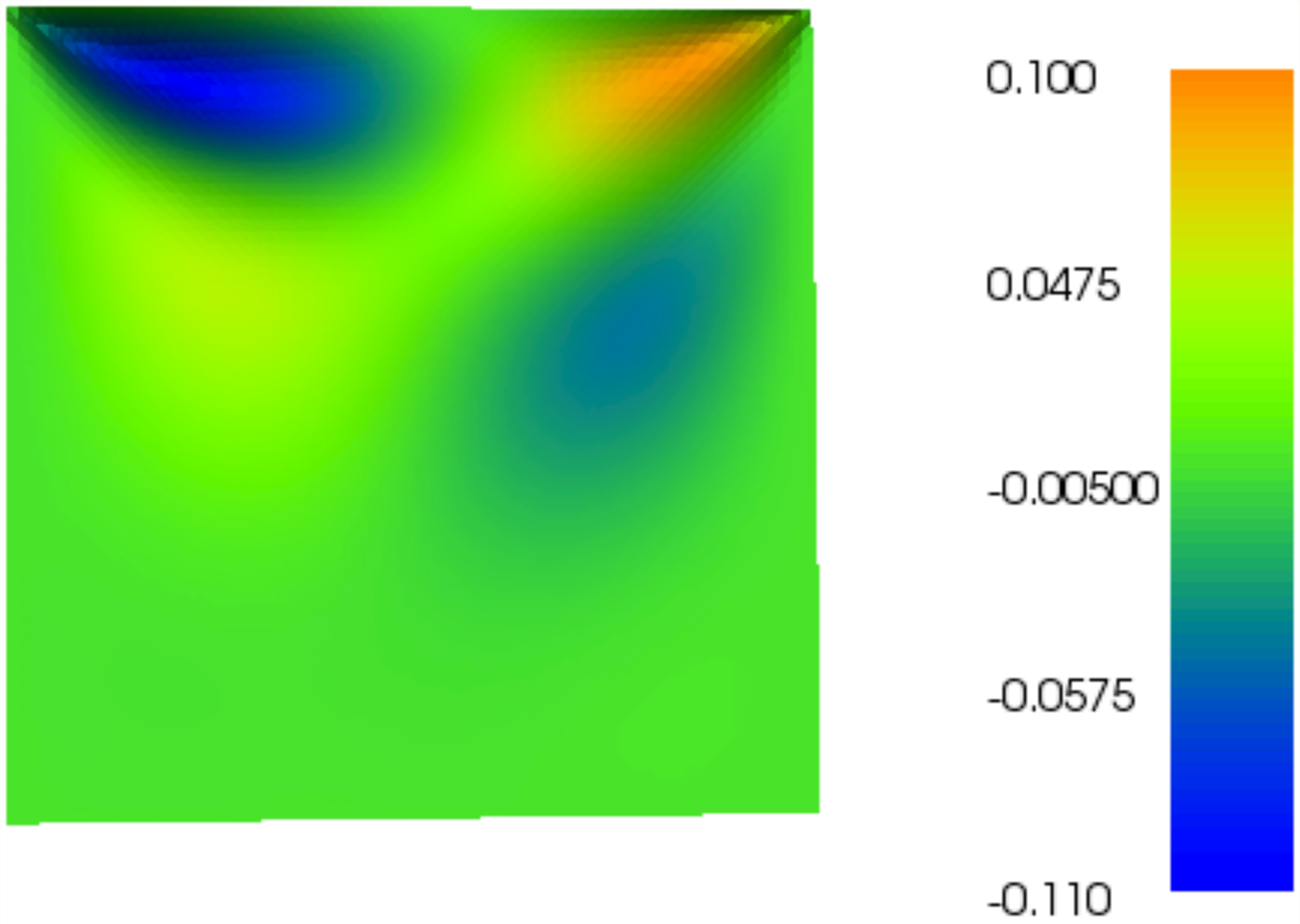

Figure 1: Driven cavity problem computed with quartics, horizontal component only. Difference between Navier-Stokes (Re=50) and Stokes (Re=0).

Driven cavity for Navier-Stokes: $R = 200$

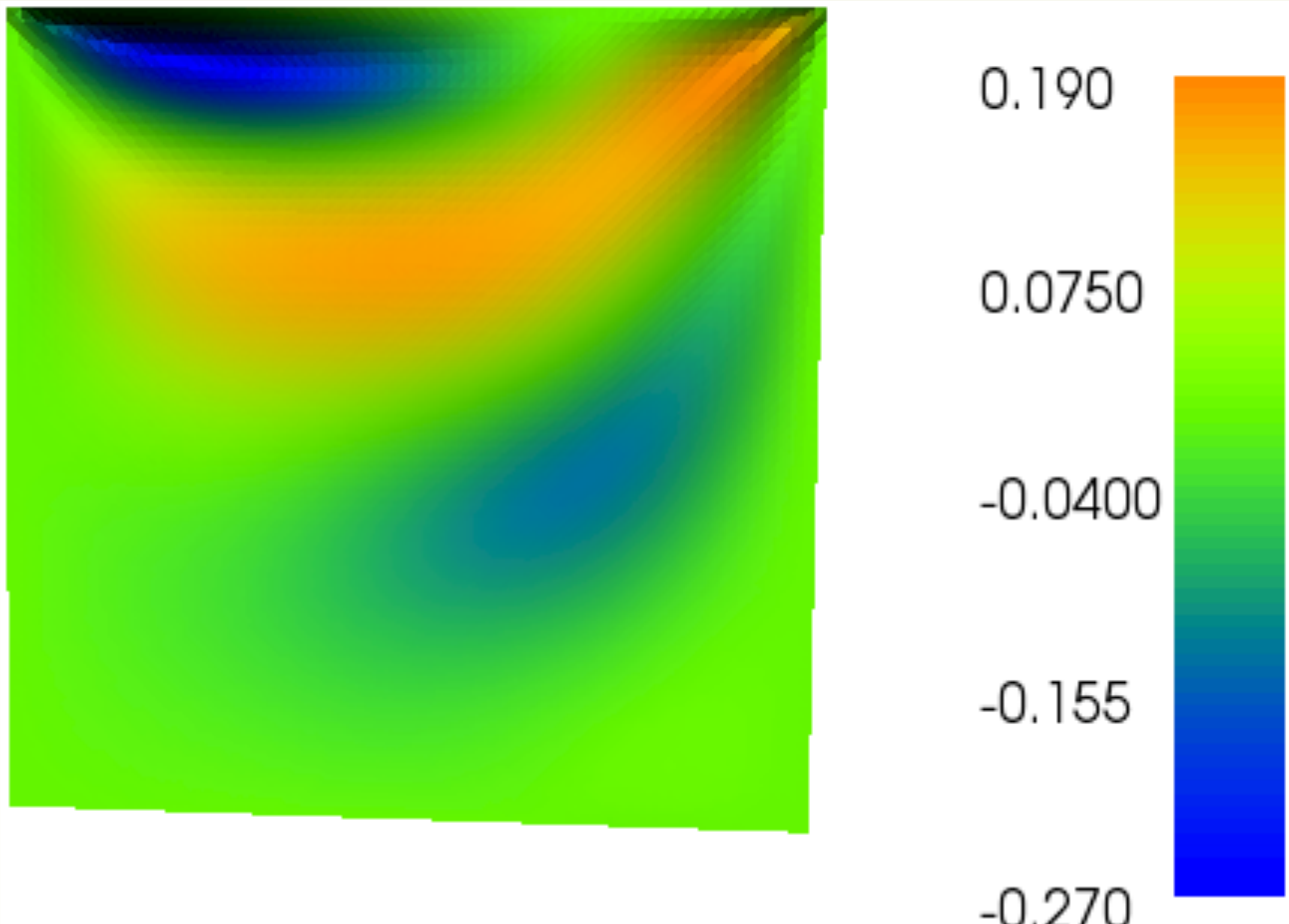

Figure 2: Driven cavity problem computed with quartics, horizontal component only. Difference between Navier-Stokes (Re=200) and Stokes (Re=0).

Driven cavity for Navier-Stokes

The solution of the driven cavity problem $\mathbf{u}_R$ is depicted in Figure 1, for two different values of R in (1).

The discretization was done with the Scott-Vogelius iterated penalty method using quartics.

Plotted is the horizontal componenet of the velocity differences $\mathbf{u}_R - \mathbf{u}_0$.

We see a substantial change in the flow pattern as a function of the Reynolds number R.

Sudden expansion

A common test problem for Navier-Stokes is a channel with a sudden expansion:

$$\Omega = \left\{ (x, y) \ : \ -L_1 \le x \le L_2, \ |y| \le \begin{cases} 1 & \text{for } x \le 0, \\ \delta^{-1} & \text{for } x \ge 0 \end{cases} \right\},$$

for some $0 < \delta < 1$.

The boundary conditions are specified by the function

$$\mathbf{g} = \begin{cases} \big((1 - y^2), 0\big) & x = -L_1, \\ \big(\delta(1 - \delta^2 y^2), 0\big) & x = L_2, \\ \mathbf{0} & \text{elsewhere}. \end{cases} \tag{28}$$

Sudden expansion

Since the flow is going from the smaller channel to the larger one, it is necessary to take $L_2 >> L_1$ to get reliable results.

It is easy to see that

$$\int_{\partial\Omega} \mathbf{g} \cdot \mathbf{n} \, ds = 0.$$

Sudden expansion: Stokes

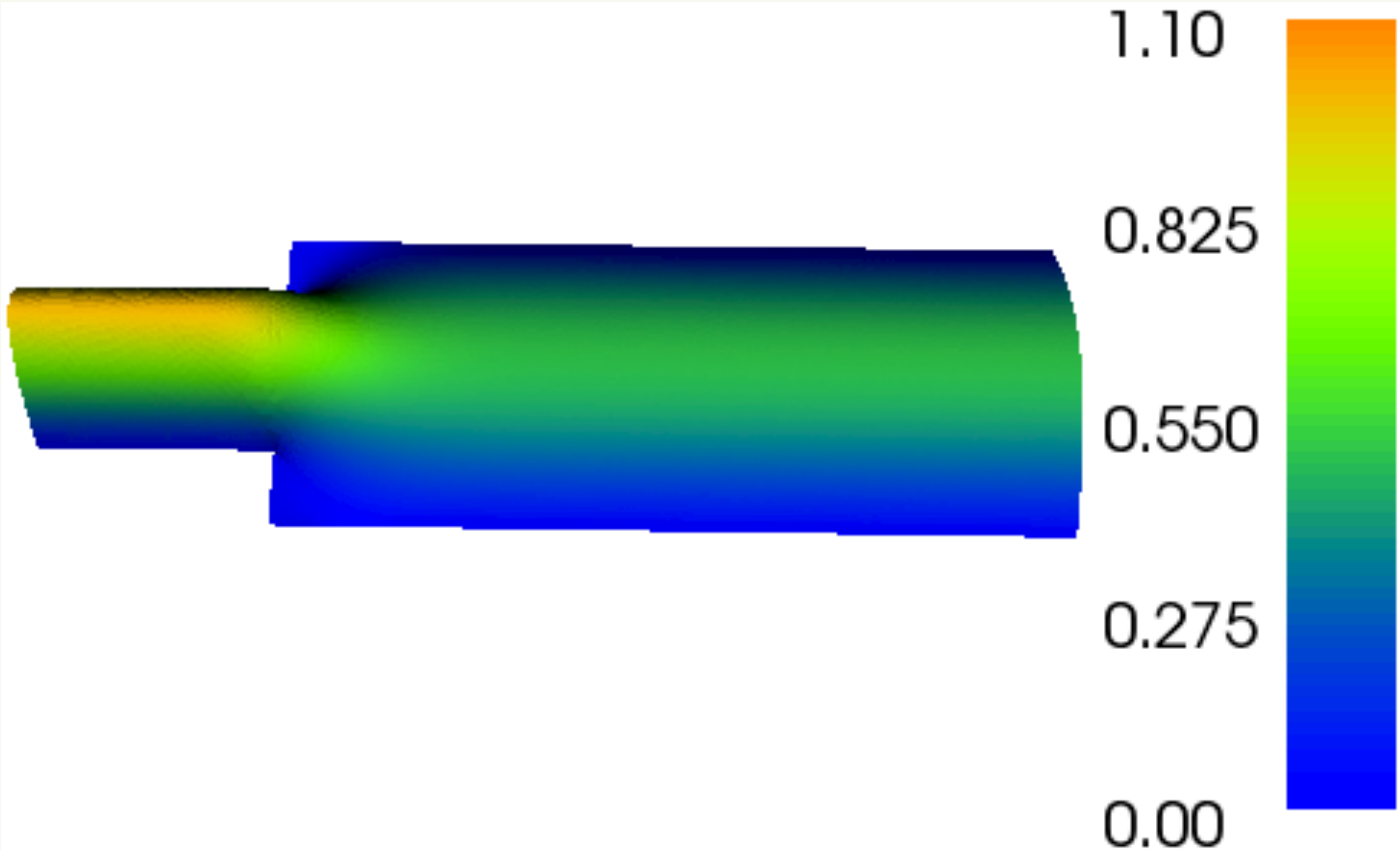

Figure 3: Sudden expansion problem computed with quartics, horizontal component only. Stokes flow.

Sudden expansion: Navier-Stokes

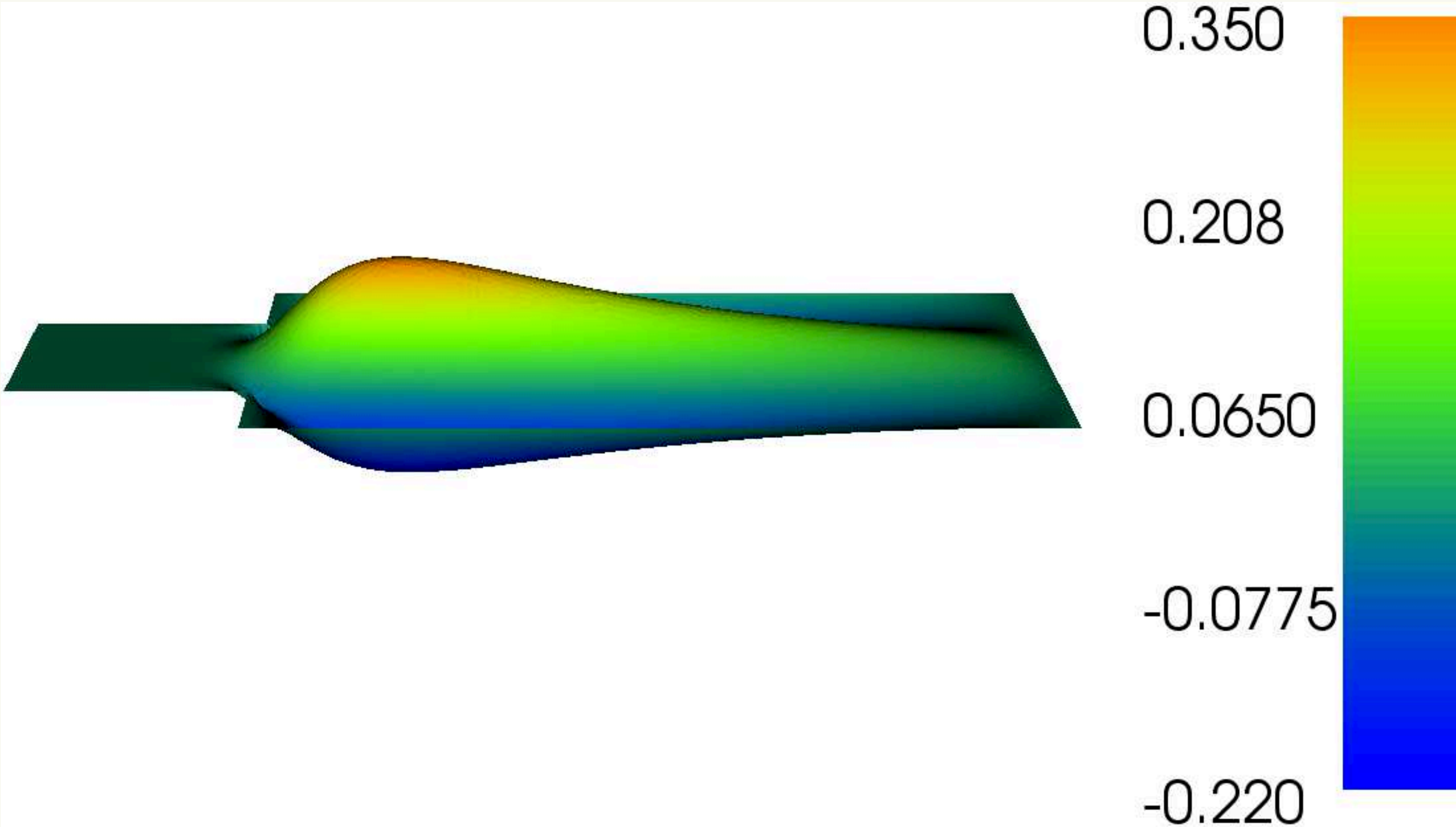

Figure 4: Sudden expansion problem computed with quartics, horizontal component only. Difference between Navier-Stokes and Stokes (Re=0) solutions for Re = 50. Domain is defined with $L_1 = 3$, $L_2 = 10$, and $\delta = \frac{1}{2}$.

Sudden expansion

The Stokes flow is symmetric in the direction orthogonal to the direction of the channel.

This can be proved by reflecting the solution around the centerline of the channel, and showing that the reflected function is also a solution.

By uniqueness of Stokes flow we conclude that the reflection must repoduce the solution.

But for Navier-Stokes flow, symmetry is not guaranteed.

It does hold for small enough Reynolds number, as indicated in Figure 5 for $R = 50$.

Sudden expansion: $R = 50$

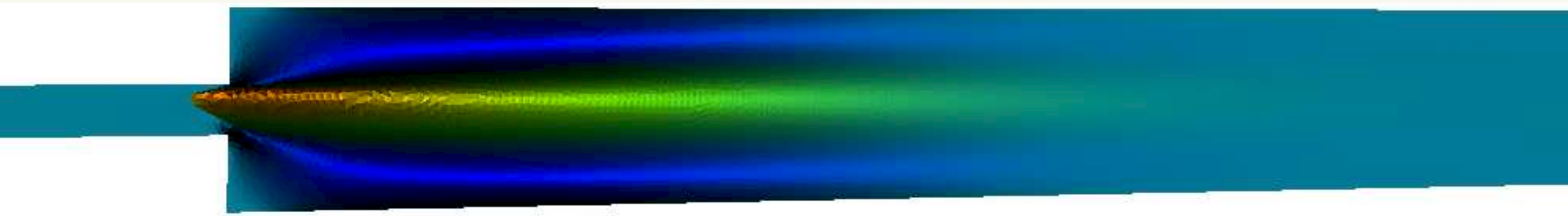

Figure 5: Sudden expansion problem computed with quartics, horizontal component only. Difference between the Navier-Stokes and Stokes (Re=0) solutions. The domain is defined with $L_1 = 10$, $L_2 = 60$, and $\delta = \frac{1}{4}$: $R = 50$.

Sudden expansion: $R = 100$

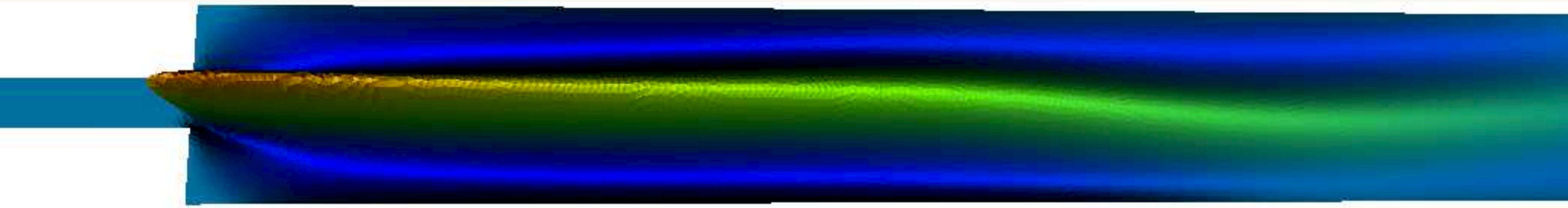

Figure 6: Sudden expansion problem computed with quartics, horizontal component only. Sudden expansion problem computed with quartics, horizontal component only. Difference between the Navier-Stokes and Stokes (Re=0) solutions. The domain is defined with $L_1 = 10$, $L_2 = 60$, and $\delta = \frac{1}{4}$: $R = 100$.

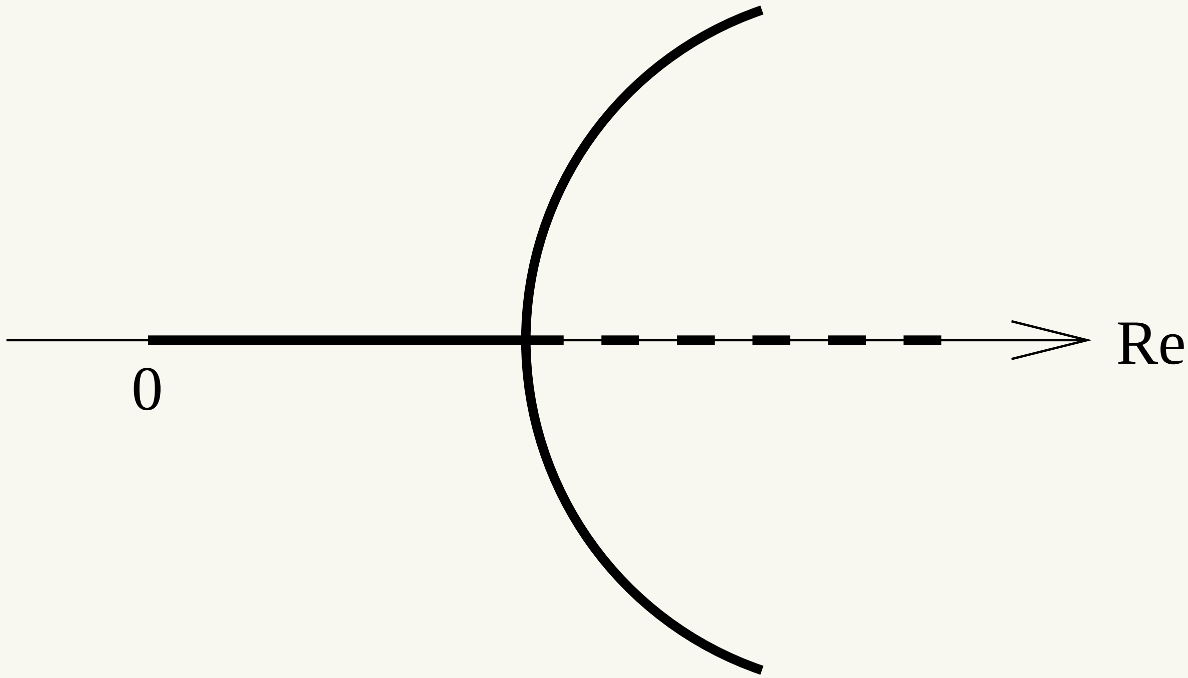

Figure 7: Idealization of the pitchfork bifurcation in the sudden-expansion problem for the Navier-Stokes equations. The dark solid lines indicate stable solutions of the Navier-Stokes equations as a function of the Reynolds number (Re). The dark dashed line indicates an unstable, symmetric solution of the Navier-Stokes equations. The unique solution indicated before the bifurcation is also symmetric.

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