Solving PDE's with FEniCS

Eigenvalues and Eigenvectors

Chapter 26

Introduction to

Automated Modeling

with FEniCS

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Oscillations

Have seen two PDE systems, wave equation and elasticity, with governing equations of the form

$$\mathbf{u}_{tt} + \mathbf{A}\mathbf{u} = \mathbf{f}, \tag{1}$$

where $\mathbf{u}$ is a vector in the discretized approximation.

Solutions oscillate as function of t.

Amplitude of oscillations can depend strongly on frequency of oscillation of forcing function $\mathbf{f}$.

Such systems exhibit **resonance**: amplitude becomes arbitrarily large as the frequency of $\mathbf{f}$ is tuned to characteristic frequencies of the system (1).

Resonance

Resonance is a phenomenon that children encounter when playing on a swing.

They learn to tune their forcing (or pumping) to a natural frequency of the swing set.

Fortunately, there is a simple algorithm for doing this, although some people are better at it than others.

Basically, you pump at the same place in the cycle every time, and this makes f repeat at the natural frequency of the swing.

Resonance is a critical factor in engineering design.

Resonance effects

Resonance can lead to disasters such as the collapse of the original Tacoma Narrows bridge [1].

Thus it is considered in the design of all large structures: bridges, buildings, etc.

Resonance is also a major factor in the amplification of the effects of earthquakes [11, 6].

Amplitude of tidal motion determined by resonance.

Resonance critical in musical instruments design.

Harmonics for string instruments are resonance.

Scalar resonance

To begin to understand resonance quantitatively, we simplify (1) even further by considering a scalar equation of the form

$$u'' + \mu u = f, \tag{2}$$

where $f(t) = \cos(\omega t)$ and ω is a constant.

We can solve (1) as $u(t) = a\cos(\omega t)$, since for this function

$$u''(t) = -a\,\omega^2 \cos(\omega t) = -\omega^2 u(t).$$

Thus

$$\begin{aligned} u''(t) + \mu u(t) &= (\mu - \omega^2)u(t) \\ &= a(\mu - \omega^2)\cos(\omega t) = a(\mu - \omega^2)f(t). \end{aligned} \tag{3}$$

Therefore $u(t) = a\cos(\omega t)$ solves (1) if

$$a = \frac{1}{\mu - \omega^2}. \tag{4}$$

The fact that the amplitude a of u goes to infinity as $\omega \to \sqrt{\mu}$ is called **resonance**.

As forcing frequency approaches critical value $\sqrt{\mu}$, solution gets arbitrarily big, even though f stays same.

Oscillations

Can solve (1) when $\mu = \omega^2$, although (3) no longer valid.

Instead, we find (Exercise 0.1) a solution

$$u(t) = t\sin(\omega t),$$

which grows without bound as time progresses.

Resonance in disasters: physical system exceeds its design limits, and then it fails.

Must avoid forcing frequency matching this critical value.

For a single equation, easy to see what critical value is, but for systems of equations, more notation needed.

Vector resonance

To solve (1) with $\mathbf{f}(t) = \mathbf{g}\cos(\omega t)$, let us try $\mathbf{u}(t) = \mathbf{a}\cos(\omega t)$ for some vector $\mathbf{a}$ to be determined.

Following the previous steps, we find

$$\mathbf{u}_{tt}(t) + \mathbf{A}\mathbf{u}(t) = (\mathbf{A} - \omega^2\mathcal{I})\mathbf{u}(t) = (\mathbf{A} - \omega^2\mathcal{I})\mathbf{a}\cos(\omega t), \quad (5)$$

where $\mathcal{I}$ is the identity matrix.

Thus we have a solution provided

$$(\mathbf{A} - \omega^2\mathcal{I})\mathbf{a} = \mathbf{g}. \quad (6)$$

So to understand resonance in this case, we need to know how big $\mathbf{a}$ can be for different values of ω.

Eigenvalues

In particular, we want to avoid situations where $\mathbf{A} - \omega^2 \mathcal{I}$ is nearly singular.

This leads us to the study of **eigenvalues**.

Eigenvalues can take various forms. The simplest context is in terms of matrices.

The eigenvalue and eigenvector pair $(\lambda, \mathbf{X})$ for a matrix $\mathbf{A}$ is defined by the equation

$$\mathbf{A}\mathbf{X} = \lambda \mathbf{X}. \tag{7}$$

The fine print requires $\mathbf{X} \neq \mathbf{0}$, for otherwise $(\lambda, \mathbf{0})$ would always be a solution for any λ.

But moreover, $\mathbf{X}$ is not unique: if $\mathbf{X}$ is a solution so is $c\mathbf{X}$ for any scalar c.

So we have to think of $\mathbf{X}$ as only prescribing a direction, not a magnitude.

Eigenvalues are ideal for answering the questions raised about resonance in (6).

If ω^2 not near an eigenvalue, then no resonance.

This is why we determine the eigenvalues of systems.

Even if we are only interested in real-valued matrices, the eigenpair $(\lambda, \mathbf{X})$ may be complex.

However, if $\mathbf{A}$ is symmetric ($\mathbf{A}^t = \mathbf{A}$) then both are real.

As we have seen, there are many applications for which $\mathbf{A}$ is symmetric, so we focus primarily on such systems.

Variational eigenproblems

Variational problems have eigenvalues too. They take the form

$$a(u,v) = \lambda\,(u,v)_{L^2(\Omega)} \quad \forall\, v \in V. \tag{8}$$

Here u is the eigenvector and λ is the eigenvalue.

Again, u is defined only up to a constant multiple, and so it is natural to constrain it in some way.

The variational formulation is appropriate for the Rayleigh[1] characterization of eigenvalues.

[1] John William Strutt, 3rd Baron Rayleigh (1842–1919) was a student of Stokes.

Define the Rayleigh quotient

$$\mathcal{R}(v) = \frac{a(v,v)}{(v,v)_{L^2(\Omega)}}, \quad v \in V, \tag{9}$$

where V is one of our finite element spaces.

In particular, eigenvalues are critical points of $\mathcal{R}$.

For example, the smallest eigenvalue is given by

$$\lambda_{\min} = \min_{0 \neq v \in V} \mathcal{R}(v). \tag{10}$$

Rayleigh quotient homogeneity

Here the notation $0 \neq v \in V$ means that $v \in V$ is not identically zero, so in particular $\|v\|_{L^2(\Omega)} > 0$.

Note that $\mathcal{R}(cv) = \mathcal{R}(v)$ for any scalar c, due to the homogeneity of $\mathcal{R}$ (it is quadratic in both the numerator and denominator).

Thus it is equivalent to write

$$\lambda_{\min} = \min_{v \in V,\ \|v\|_{L^2(\Omega)}=1} \mathcal{R}(v). \tag{11}$$

Rayleigh quotient as a lower bound

Writing out the Rayleigh quotient, we have

$$\lambda_{\min} = \min_{0\neq v\in V} \frac{a(v,v)}{(v,v)_{L^2(\Omega)}} = \min_{0\neq v\in V} \frac{a(v,v)}{\|v\|^2_{L^2(\Omega)}}.$$

Thus

$$\lambda_{\min} \leq \frac{a(v,v)}{\|v\|^2_{L^2(\Omega)}} \quad \forall\, 0 \neq v \in V.$$

Multiplying by $\|v\|_{L^2(\Omega)}$, we find

$$\lambda_{\min}\|v\|^2_{L^2(\Omega)} \leq a(v,v) \quad \forall\, v \in V. \tag{12}$$

Note that this holds trivially for $v \equiv 0$.

Rayleigh quotient and coercivity

We recognize (12) as a statement of coercivity if $\lambda_{\min} > 0$.

This demonstrates a theoretical role for eigenvalues, in addition to the phenomenological role with regard to resonance discussed earlier.

One simple algorithm for approximating eigenvalues is **Rayleigh quotient iteration**:

$$
\begin{aligned}
a(w^k, v) - \lambda^k (w^k, v)_{L^2(\Omega)} &= (u^k, v)_{L^2(\Omega)} \quad \forall\, v \in V, \\
u^{k+1} &= \|w^k\|^{-1}_{L^2(\Omega)}\, w^k, \\
\lambda^{k+1} &= \mathcal{R}\big(u^{k+1}\big).
\end{aligned}
\tag{13}
$$

Note that

$$\|u^{k+1}\|_{L^2(\Omega)} = \|w^k\|^{-1}_{L^2(\Omega)} \|w^k\|_{L^2(\Omega)} = 1,$$

so that $\mathcal{R}\big(u^{k+1}\big) = a(u^{k+1}, u^{k+1})$.

Need to start with some u^0 and λ^0; quality of guesses will affect efficiency of the algorithm.

Algorithm details

Some care needs to be taken regarding solving for w^k since the system used is approaching singularity.

However, this is part of the magic of the algorithm [9, Section 15.2.1].

If we know that $a(v, v) \geq 0$ for all $v \in V$, then we could take any $\lambda^0 < 0$ to approximate the lowest eigenvalue.

For the lowest eigenvalue, the corresponding eigenvector is often nonnegative, so that information could also be used.

For example, we might take $u^0 \equiv 1$.

Characteristic values

The eigenvalues of a given operator, such as the Laplacian, will depend on the domain.

Moreover, they can be characteristic of the domain, allowing the identification of the domain from the eigenvalues [5].

The word “eigen” is German for “characteristic,” and one thus often sees the term **characteristic values** instead of eigenvalues.

Review of the Helmholtz problem

The Helmholtz problem involves solving

$$-k^2 u(\mathbf{x}) - \Delta u(\mathbf{x}) = f(\mathbf{x}). \tag{14}$$

But we know that there can be eigenvalues $\lambda = k^2$ where the corresponding eigenvector u_λ satisfies

$$-\Delta u_\lambda(\mathbf{x}) = \lambda u_\lambda(\mathbf{x}), \tag{15}$$

leading to a null solution u_λ of (14).

Expected resonance

For k near $\sqrt{\lambda}$ we thus expect a resonance phenomenon, and if $\lambda = k^2$ the character of the solution can change completely.

In particular, f must satisfy constraints.

If λ is a simple eigenvalue [10], then the constraint is that $(f, u_\lambda)_{L^2(\Omega)} = 0$.

Meaning of Gårding's inequality

Gårding's inequality (10.21) can be interpreted as follows. Let $\rho = \gamma_2/\gamma_1$. Then dividing by γ_1, (10.21) can be rewritten

$$\gamma_1^{-1}\|v\|_{H^1(\Omega)}^2 \le a(v,v) + \rho\|v\|_{L^2(\Omega)}^2 \quad \forall v \in V.$$

This is the coercivity inequality for the new bilinear form

$$a_\rho(u,v) := a(u,v) + \rho(u,v)_{L^2(\Omega)} \quad \forall u,v \in V.$$

If λ is an eigenvalue for $a_\rho(\cdot,\cdot)$, then $\lambda - \rho$ is an eigenvalue for $a(\cdot,\cdot)$ (Exercise 0.4). Since coercivity is a statement about positivity of eigenvalues, this means that the eigenvalues of $a(\cdot,\cdot)$ are bounded below by $-\rho$.

Estimating the inf-sup constant

Several papers have addressed the issue of estimating computationally the inf-sup constant for various spaces and variational problems [7, 8, 2].

We show here that this can be thought of as an eigenvalue problem [4].

Define κ by

$$\kappa = \min_{0 \neq v \in V_h,\, v \perp_a Z_h} \frac{(\nabla\cdot v, \nabla\cdot v)_{L^2}}{a(v,v)} = \min_{0 \neq v \in Z_h^{\perp}} \frac{(\nabla\cdot v, \nabla\cdot v)_{L^2}}{a(v,v)}, \tag{16}$$

where $Z_h^{\perp} = \{\mathbf{v} \in V_h \,:\, a(\mathbf{v},\mathbf{w}) = 0 \;\forall \mathbf{w} \in Z_h\}$.

We have $\beta \geq \sqrt{\kappa}$, where β is the constant in the inf-sup condition provided we define $\|v\|_V = \sqrt{a(v,v)}$.

LEMMA: Suppose $\|v\|_V = \sqrt{a(v,v)}$ and $Q_h = \nabla\cdot V_h$. Then

$$
\begin{aligned}
\beta &= \inf_{0\neq q\in Q_h} \sup_{0\neq v\in V_h} \frac{b(v,q)}{\|v\|_V\|q\|_{L^2}} \\
&= \inf_{0\neq q\in Q_h} \sup_{0\neq v\in Z_h^{\perp}} \frac{b(v,q)}{\|v\|_V\|q\|_{L^2}} \geq \sqrt{\kappa},
\end{aligned}
\tag{17}
$$

where κ is defined in (16).

The proof

Proof. Take $q = \nabla\cdot w$ where $w \in Z_h^\perp$. Then

$$\inf_{0\neq w\in Z_h^\perp} \sup_{0\neq v\in Z_h^\perp} \frac{(\nabla\cdot v, \nabla\cdot w)_{L^2}}{\|\nabla\cdot w\|_{L^2}\|v\|_V} \geq \inf_{0\neq w\in Z_h^\perp} \frac{(\nabla\cdot w, \nabla\cdot w)_{L^2}}{\|\nabla\cdot w\|_{L^2}\|w\|_V} = \inf_{0\neq w\in Z_h^\perp} \frac{\|\nabla\cdot w\|_{L^2}}{\|w\|_V} = \sqrt{\gamma} \tag{18}$$

by taking $v = w$.

Converse lemma

On the other hand, [3, (13.1.16)] implies that

$$\sqrt{\kappa} \geq \beta\Big(\frac{\alpha}{\alpha + C_a}\Big).$$

With the choice of norm $\|v\|_V = \sqrt{a(v,v)}$, then $C_a = \alpha = 1$.

Thus β and $\sqrt{\kappa}$ are essentially equivalent parameters measuring the stability of the Stokes approximation.

An eigenproblem

Consider the eigenproblem: find $0 \neq \mathbf{u}_h \in Z_h^\perp$ such that

$$(\nabla \cdot \mathbf{u}_h, \nabla \cdot \mathbf{v}) = \lambda a(\mathbf{u}_h, \mathbf{v}) \; \forall \mathbf{v} \in Z_h^\perp.$$

Let $\lambda_{\min}$ be the smallest eigenvalue, which is given by the Rayleigh quotient

$$\lambda_{\min} = \min_{\mathbf{v} \in Z_h^\perp} \frac{(\nabla \cdot \mathbf{v}, \nabla \cdot \mathbf{v})}{a(\mathbf{v}, \mathbf{v})} > 0. \tag{19}$$

Note that $\lambda_{\min} > 0$ since $\lambda = 0$ leads to the contradiction $\nabla \cdot \mathbf{u}_h = 0$, that is, $\mathbf{u}_h \in Z_h \cap Z_h^\perp$.

Now solve for $\mathbf{u}_h$ via Rayleigh quotient iteration (**RQI**).

Solve by RQI

Find $\mathbf{u}^k \in Z_h^\perp$ such that

$$\begin{aligned}(\nabla\cdot \mathbf{u}^{k-1}, \nabla\cdot \mathbf{v}) &= \lambda^k a(\mathbf{u}^k, \mathbf{v}) \;\forall \mathbf{v} \in Z_h^\perp \\ \lambda^{k+1} &= \frac{(\nabla\cdot \mathbf{u}^k, \nabla\cdot \mathbf{u}^k)}{a(\mathbf{u}^k, \mathbf{u}^k)} \geq \lambda_{\min}.\end{aligned} \tag{20}$$

Now we consider how to compute this despite the fact that the space $Z_h^\perp$ is not explicitly known.

In the case that $a(\cdot,\cdot)$ is coercive on all of V_h, we can implement (20) by solving

$$a(\mathbf{u}^k, \mathbf{v}) = (\lambda^k)^{-1}(\nabla\cdot \mathbf{u}^{k-1}, \nabla\cdot \mathbf{v}) \;\forall \mathbf{v} \in V_h.$$

Then $a(\mathbf{u}^k, \mathbf{v}) = 0$ for all $\mathbf{v} \in Z_h$, so $\mathbf{u}^k \in Z_h^\perp$ for all $k > 0$.

Comparison with earlier techniques

The approach advocated in [7, 8, 2] requires working with a larger space than $\nabla\cdot V_h$, and it thus finds spurious modes in addition to estimating the constant $\kappa = \lambda_{\min}$.

By contrast, our approach has no spurious modes, since $\kappa = \lambda_{\min} > 0$, and it identifies concretely an approximation of the extrema of (16).

Thus there is a fundamental philosophical difference regarding spurious modes.

Using our approach, the concept of spurious modes itself becomes spurious.

Exercise 0.1 *Consider (2) in the case that $\omega = \sqrt{\mu}$. In this case, the formula (3) is no longer valid. Show that instead, there is a solution of the form*

$$u(t) = \frac{t}{2\omega} \sin(\omega t).$$

(Hint: write $u = tv$ and show that $u'' = 2v' + tv''$. Be sure to say what the formula for v is. Next show that $tv'' = -\omega^2 u$, so that $u'' + \omega^2 u = 2v'$. Finally, verify that $2v' = \cos(\omega t)$.)

Exercise 0.2 *Consider the Laplacian on the square with homogeneous Dirichlet conditions. Show that it has an eigenvalue* $2\pi^2$ *with corresponding eigenvector* $u(x,y)=(\sin\pi x)(\sin\pi y)$*.*

Exercise 0.3 *Consider the Laplacian on the square. Approximate its lowest eigenvalue using the Rayleigh quotient iteration starting with* $\lambda^0=0$ *and* $u^0\equiv 1$*. Compare your answer with* $\lambda=2\pi^2$*. Do you get close if you take the degree of approximation high enough and the mesh fine enough?*

Exercise 0.4 *Suppose that $a(\cdot,\cdot)$ is a bilinear form on a space V and $\rho \in \mathbb{R}$. Define a new bilinear form*

$$a_\rho(u,v) := a(u,v) + \rho(u,v)_{L^2(\Omega)} \quad \forall u,v \in V.$$

Suppose λ is an eigenvalue for $a_\rho(\cdot,\cdot)$. Show that $\lambda - \rho$ is an eigenvalue for $a(\cdot,\cdot)$

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