Solving PDE's with FEniCS

Elasticity

Chapter 19

Introduction to

Automated Modeling

with FEniCS

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Solid-fluid Analogy

There is a strong analogy between models for fluids and solids.

The model equations for all solids take the form

$$\rho \mathbf{u}_{tt} = \nabla \cdot \mathbf{T} + \mathbf{f},$$

where $\mathbf{u}$ is the displacement of the solid, $\mathbf{T}$ is called the Cauchy stress and $\mathbf{f}$ is externally given data.

The divergence operator on a matrix function is defined by

$$(\nabla \cdot \mathbf{T})_i = \sum_{j=1}^{d} T_{ij,j}$$

The models differ based on the way the stress $\mathbf{T}$ depends on the displacement $\mathbf{u}$.

Time-independent models take the form

$$-\nabla\cdot\mathbf{T}=\mathbf{f}. \tag{1}$$

The simplest expression for the stress is linear:

$$\mathbf{T}=\mathbf{C}:\boldsymbol{\epsilon},$$

where $\mathbf{C}$ is a material tensor, the **constituitive matrix**, and $\boldsymbol{\epsilon}=\frac{1}{2}\big(\nabla\mathbf{u}+\nabla\mathbf{u}^t\big)$.

Such solids are called elastic.

For isotropic models,

$$C_{ijkl} = K\delta_{ij}\delta_{kl} + \mu\Big(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk} - \tfrac{2}{3}\delta_{ij}\delta_{kl}\Big), \quad \text{for } i,j,k,l = 1,2,3, \tag{2}$$

where δ_{ij} is the Kronecker-δ, K is the **bulk modulus** (or incompressibility) and μ is the **shear modulus**.

The tensor contraction $\mathbf{C} : \boldsymbol{\epsilon}$ is defined by

$$(\mathbf{C} : \boldsymbol{\epsilon})_{ij} = \sum_{kl=1}^{d} C_{ijkl}\epsilon_{kl}.$$

Elasticity variational formulation

Carrying out the tensor contraction $\mathbf{T} = \mathbf{C} : \boldsymbol{\epsilon}$, we have

$$\begin{aligned} T_{ij} &= K\delta_{ij}\epsilon_{kk} + 2\mu\big(\epsilon_{ij} - \tfrac{1}{3}\delta_{ij}\epsilon_{kk}\big) = \lambda\delta_{ij}\epsilon_{kk} + 2\mu\epsilon_{ij} \\ &= \lambda\delta_{ij}\nabla\cdot\mathbf{u} + \mu\big(\nabla\mathbf{u} + \nabla\mathbf{u}^t\big)_{ij} \\ &= \lambda\delta_{ij}\nabla\cdot\mathbf{u} + \mu\big(u_{i,j} + u_{j,i}\big), \end{aligned} \tag{3}$$

where $\lambda(= K - \frac{2}{3}\mu)$ and μ are known as the

Lamé parameters

and the Einstein summation convention was used, e.g.,

$$\epsilon_{kk} = \sum_{k=1}^{3} \epsilon_{kk} = \nabla\cdot\mathbf{u}.$$

The variational formulation of (1) takes the form:

Find $\mathbf{u} \in V + \boldsymbol{\gamma}$ such that

$$a_C(\mathbf{u}, \mathbf{v}) = F(\mathbf{v}) \quad \forall \mathbf{v} \in V\,, \tag{4}$$

where $a_C(\cdot, \cdot)$ and $F(\cdot)$ are given by

$$\begin{aligned} a_C(\mathbf{u}, \mathbf{v}) &:= \int_\Omega \mathbf{T} : \nabla \mathbf{v}\, d\mathbf{x} \\ &= \lambda \int_\Omega (\nabla \cdot \mathbf{u})(\nabla \cdot \mathbf{v})\, d\mathbf{x} + \mu \int_\Omega \left(\nabla \mathbf{u} + \nabla \mathbf{u}^t\right) : \nabla \mathbf{v}\, d\mathbf{x}, \end{aligned} \tag{5}$$

and

$$F(\mathbf{v}) := \int_\Omega \mathbf{f} \cdot \mathbf{v}\, d\mathbf{x}. \tag{6}$$

Elasticity variational formulation

Derivation: multiply (3) by $\mathbf{v}$ with a "dot" product, integrate by parts.

The space V consists of the d-fold Cartesian product of the subset of $H^1(\Omega)$ of functions vanishing on the boundary.

Now let us consider some special cases.

Several cases arise where the three-dimensional problem has a symmetry that reduces the model to a two-dimensional one, as follows.

A dimensional reduction

Idealized state when dimension of Ω is large in $z = x_3$-direction.

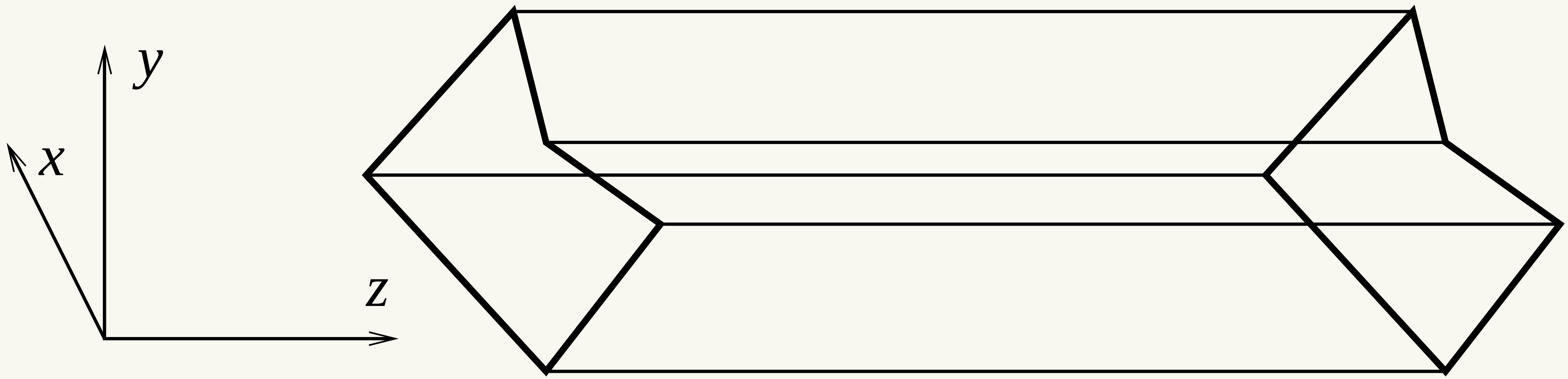


Figure 1: A two-dimensional reduction. $\Omega = \widehat{\Omega} \times [0, Z]$.

Two applications:

- Anti-plane strain
- Plane strain

Anti-plane strain

In anti-plane strain [4], component of strain normal to a (x_1, x_2) plane is the only non-zero displacement, that is, $u_1 = u_2 = 0$, and thus $\mathbf{u} = (0, 0, w)$.

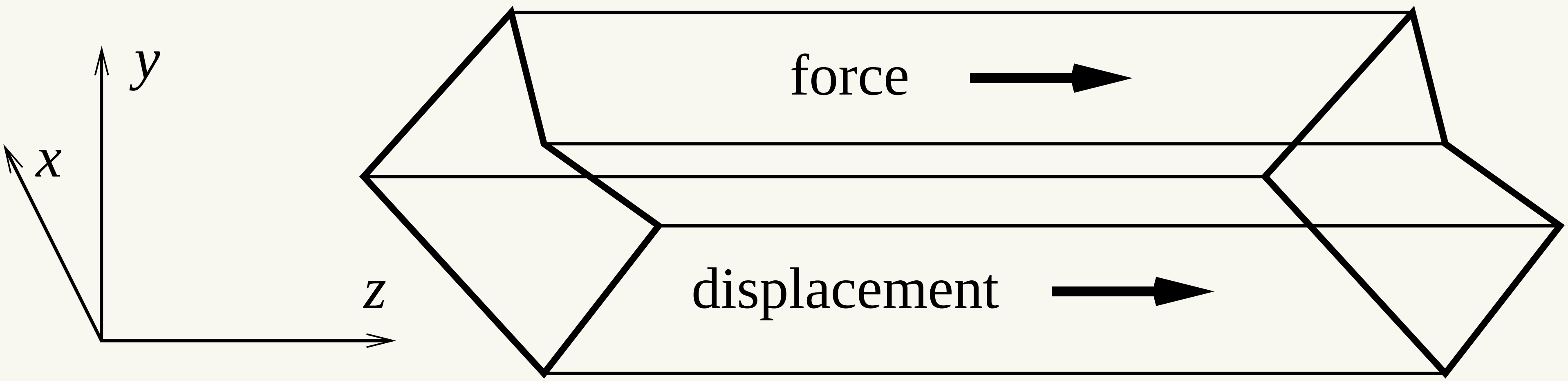


Figure 2: Anti-plane strain

The applied force is in that direction only, that is, $\mathbf{f} = (0, 0, f)$.

Anti-plane strain is Laplace

It is assumed that the displacement $w = u_3$ is independent of x_3, although it does depend on (x_1, x_2).

In particular, $\nabla \cdot \mathbf{u} = 0$.
Thus

$$\nabla \mathbf{u} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ w_{,1} & w_{,2} & 0 \end{pmatrix} \text{ and } \boldsymbol{\epsilon} = \tfrac{1}{2} \begin{pmatrix} 0 & 0 & w_{,1} \\ 0 & 0 & w_{,2} \\ w_{,1} & w_{,2} & 0 \end{pmatrix}.$$

Therefore

$$\mathbf{T} = \mu \begin{pmatrix} 0 & 0 & w_{,1} \\ 0 & 0 & w_{,2} \\ w_{,1} & w_{,2} & 0 \end{pmatrix}.$$

Anti-plane strain is Laplace

Thus

$$\nabla\cdot \mathbf{T} = \mu \begin{pmatrix} 0 \\ 0 \\ w_{,11} + w_{,22} \end{pmatrix} = \mu \begin{pmatrix} 0 \\ 0 \\ \Delta w \end{pmatrix}.$$

Therefore (1) becomes

$$-\mu \begin{pmatrix} 0 \\ 0 \\ \Delta w \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ f \end{pmatrix}.$$

Thus anti-plane strain reduces to equation,

$$-\mu \Delta w = f.$$

So our many techniques for the Laplace equation apply.

Plane strain

In plane strain [4], component of strain normal to (x, y) plane) is zero.

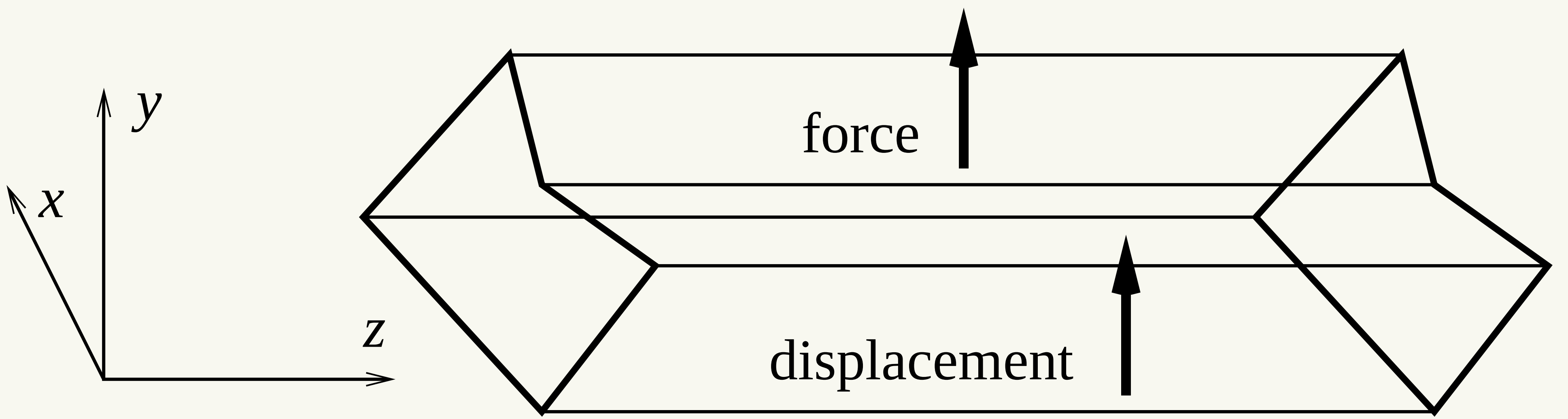


Figure 3: A two-dimensional reduction. $\Omega = \widehat{\Omega} \times [0, Z]$.

Applied forces in the z-direction are zero.

Plane strain

Thus $\mathbf{u} = (u, v, 0)$, $\nabla \mathbf{u} = \begin{pmatrix} u_{,x} & u_{,y} & 0 \\ v_{,x} & v_{,y} & 0 \\ 0 & 0 & 0 \end{pmatrix}$ and

$$\boldsymbol{\epsilon} = \begin{pmatrix} u_{,x} & \frac{1}{2}(u_{,y} + v_{,x}) & 0 \\ \frac{1}{2}(u_{,y} + v_{,x}) & v_{,y} & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Thus variational problem (4) applies where integration in (5) and (6) over two-dimensional domain.

Plane strain variational form

In particular,

$$\mathbf{T} = \lambda \nabla\cdot \mathbf{u} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \mu \begin{pmatrix} u_{,x} & \frac{1}{2}(u_{,y} + v_{,x}) & 0 \\ \frac{1}{2}(u_{,y} + v_{,x}) & v_{,y} & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

where $\nabla\cdot \mathbf{u} = u_{,x} + v_{,y}$.

Choosing test functions $\mathbf{v} = (v_1, v_2, 0)$ in (5), we reduce to a two-dimensional problem of the same form as (5).

Plane strain variational formulation

Find $\mathbf{u}$ such that $\mathbf{u}-\boldsymbol{\gamma} \in V$ such that

$$a_C\left(\mathbf{u},\mathbf{v}\right) = F(\mathbf{v}) \quad \forall \mathbf{v} \in V\,, \tag{7}$$

where $a_C(\cdot,\cdot)$ and $F(\cdot)$ are given by

$$\begin{aligned} a_C(\mathbf{u},\mathbf{v}) := \lambda \int_\Omega (\nabla\cdot \mathbf{u})(\nabla\cdot \mathbf{v})\, d\mathbf{x} \\ + \mu \int_\Omega \left(\nabla\mathbf{u} + \nabla\mathbf{u}^t\right) : \nabla\mathbf{v}\, d\mathbf{x}, \end{aligned} \tag{8}$$

and

$$F(\mathbf{v}) := \int_\Omega \mathbf{f}\cdot\mathbf{v}\, d\mathbf{x}, \tag{9}$$

and all functions and integration are in 2-D only.

Plates are thin, planar structures.

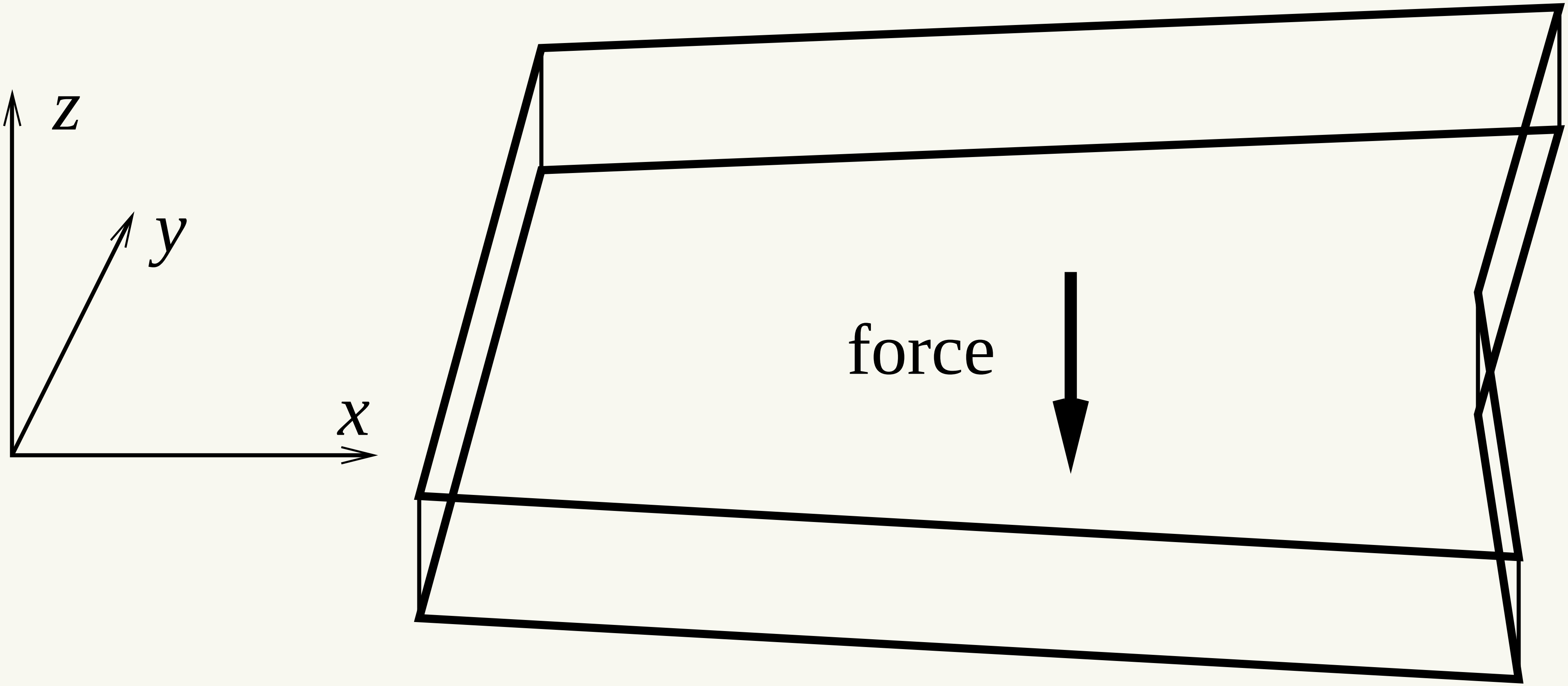


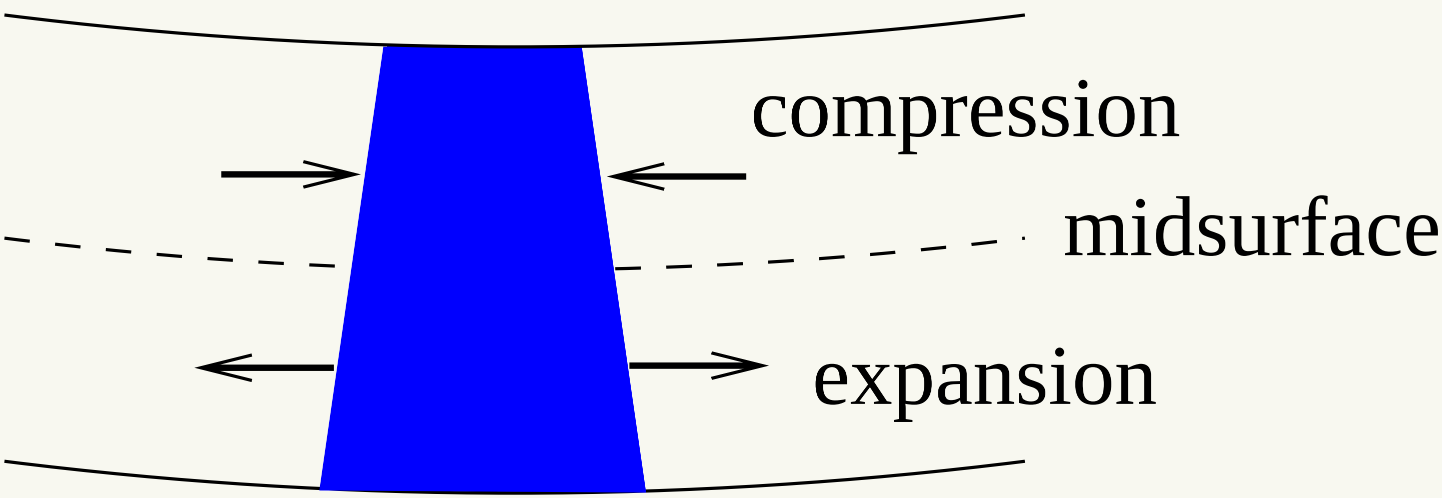
Figure 4: Another two-dimensional reduction. $\Omega = \widehat{\Omega} \times [-\tau, \tau]$.

Suppose that $\widehat{\Omega}$ is some domain in the x, y plane, and

$$\Omega = \left\{ (x, y, z) \; : \; (x, y) \in \widehat{\Omega}, \; z \in [-\tau, \tau] \right\},$$

where τ is small compared to dimensions of $\widehat{\Omega}$.

When the structure is deformed, the behavior is different on each side of the midsurface $\tau = 0$.



Using the **Kirchhoff hypothesis**, the displacement $\mathbf{u} = (u, v, w)$ satisfies

$$u \approx -zw_{,x}, \quad v \approx -zw_{,y}.$$

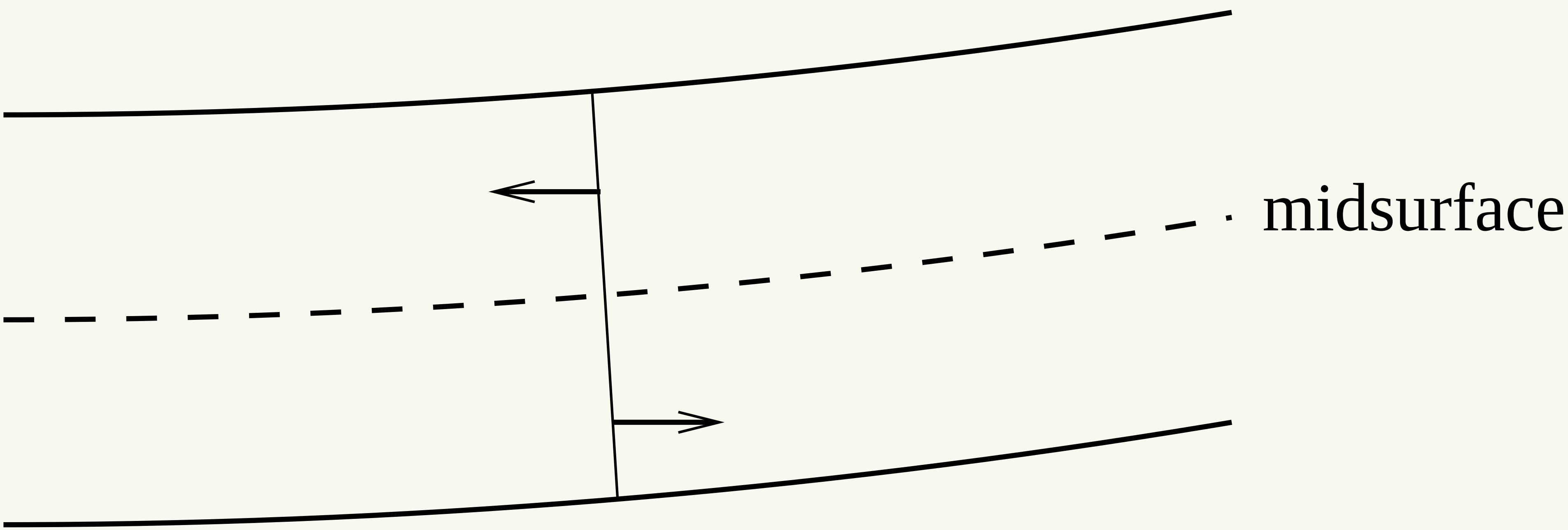


Figure 5: Relation between out-of-plane dispalcement and in-plane displacement leading to the Kirchhoff hypothesis.

Another view of the **Kirchhoff hypothesis**:

$$u \approx -z w_{,x}, \quad v \approx -z w_{,y}.$$

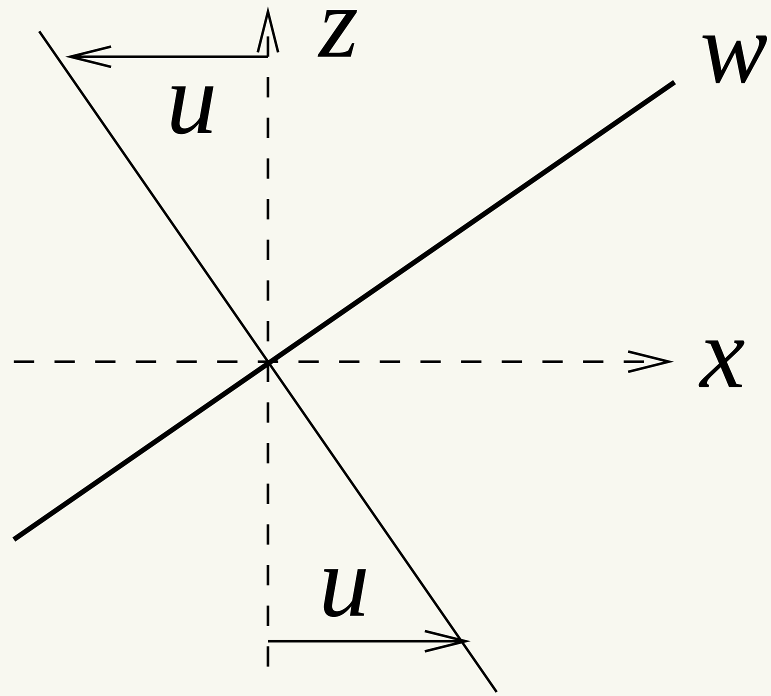


Figure 6: Relation between out-of-plane dispalcement and in-plane displacement leading to the Kirchhoff hypothesis.

Plate-Bending simplification

Equations of elasticity can be written as equation for just deflection w normal to plane of plate:

$$\nabla \mathbf{u} = \begin{pmatrix} u_{,x} & u_{,y} & u_{,z} \\ v_{,x} & v_{,y} & v_{,z} \\ w_{,x} & w_{,y} & w_{,z} \end{pmatrix} = \begin{pmatrix} -zw_{,xx} & -zw_{,xy} & -w_{,x} \\ -zw_{,xy} & -zw_{,yy} & -w_{,y} \\ w_{,x} & w_{,y} & 0 \end{pmatrix},$$

$$\boldsymbol{\epsilon} = \begin{pmatrix} -zw_{,xx} & -zw_{,xy} & 0 \\ -zw_{,xy} & -zw_{,yy} & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

and $\mathbf{T} = -z\lambda\Delta w \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \mu \begin{pmatrix} -zw_{,xx} & -zw_{,xy} & 0 \\ -zw_{,xy} & -zw_{,yy} & 0 \\ 0 & 0 & 0 \end{pmatrix}.$

Deriving plate-bending variational form

Multiplying by $\mathbf{v} = -z\nabla\phi$ where ϕ depends only on x, y, integrating over Ω and integrating by parts, we find

$$\lambda \int_\Omega z^2 \Delta w \Delta\phi \, d\mathbf{x} + \mu \int_\Omega z^2 \begin{pmatrix} w_{,xx} & w_{,xy} \\ w_{,xy} & w_{,yy} \end{pmatrix} : \begin{pmatrix} \phi_{,xx} & \phi_{,xy} \\ \phi_{,xy} & \phi_{,yy} \end{pmatrix} d\mathbf{x}$$
$$= \int_\Omega \mathbf{f} \cdot \mathbf{v} \, d\mathbf{x}.$$

For simplicity, we consider the case where the body force $\mathbf{f} = \mathbf{0}$. We can expand to get

$$\begin{pmatrix} w_{,xx} & w_{,xy} \\ w_{,xy} & w_{,yy} \end{pmatrix} : \begin{pmatrix} \phi_{,xx} & \phi_{,xy} \\ \phi_{,xy} & \phi_{,yy} \end{pmatrix} = w_{,xx}\phi_{,xx} + 2w_{,xy}\phi_{,xy} + w_{,yy}\phi_{,yy}$$
$$= \Delta w \Delta\phi - w_{,xx}\phi_{,yy} - w_{,yy}\phi_{,xx} + 2w_{,xy}\phi_{,xy}.$$

Note that the integral of z^2 with respect to z over the plate thickness is equal to $\frac{2}{3}\tau^3$.

Then the above becomes

$$\frac{2}{3}\tau^3 \int_{\widehat{\Omega}} (\lambda+\mu)\Delta w \Delta\phi + \mu(-w_{,xx}\phi_{,yy} - w_{,yy}\phi_{,xx} + 2w_{,xy}\phi_{,xy})\, dxdy$$

Let $a_P(\cdot,\cdot)$ be bilinear form defined on $H^2(\Omega)$ given by

$$\begin{aligned} a_P(u,v) :=& \int_\Omega \Delta u\, \Delta v\, dx\, dy \\ &- (1-\nu)\int_\Omega 2u_{xx}v_{yy} + 2u_{yy}v_{xx} - 4u_{xy}v_{xy}\, dx\, dy. \end{aligned} \tag{10}$$

Poisson's ratio ν

The constant ν in (10) is a physical constant known as Poisson's ratio, and $2(1-\nu) = \mu/(\lambda+\mu)$.

In the model for the bending of plates, ν is restricted to the range $[0, \frac{1}{2}]$.

However, $a_P(\cdot,\cdot)$ is known [1] to satisfy a Gårding-type inequality,

$$a_P(v,v) + K\|v\|^2_{L^2(\Omega)} \geq \alpha\|v\|^2_{H^2(\Omega)} \quad \forall v \in H^2(\Omega)\,, \qquad (11)$$

where $\alpha > 0$ and $K < \infty$, for all $-3 < \nu < 1$.

For $\nu = 1$, such an inequality cannot hold as $a_P(v,v)$ vanishes in that case for all harmonic functions, v.

Plate-bending coercivity

Coercivity estimate for $0 < \nu < 1$: Write

$$a_P(v,v) = \int_\Omega \nu\,(v_{xx}+v_{yy})^2 + (1-\nu)\Big(\,(v_{xx}-v_{yy})^2 + 4v_{xy}^2\Big)\,dxdy$$

$$\geq \min\{\nu, 1-\nu\} \int_\Omega (v_{xx}+v_{yy})^2 + (v_{xx}-v_{yy})^2 + 4v_{xy}^2\,dxdy$$

Thus

$$\begin{aligned} a_P(v,v) \geq &= 2\min\{\nu,1-\nu\} \int_\Omega v_{xx}^2 + v_{yy}^2 + 2v_{xy}^2\,dxdy \\ &= 2\min\{\nu,1-\nu\}|v|_{H^2(\Omega)}^2. \end{aligned}$$

So $a_P(\cdot,\cdot)$ is coercive over any closed subspace, $V \subset H^2(\Omega)$, such that $V \cap \mathcal{P}_1 = \emptyset$ (see [5]).

Plate-vending variational problem

Thus, there is a constant $\alpha > 0$ such that

$$a_P(v,v) \geq \alpha \|v\|^2_{H^2(\Omega)} \quad \forall v \in V. \tag{12}$$

For $F \in H^2(\Omega)'$ and $V \subset H^2(\Omega)$, consider

find $u \in V$ such that

$$a_P(u,v) = F(v) \quad \forall v \in V. \tag{13}$$

Following is consequence of Lax-Milgram Theorem.

Theorem 0.1 *If $V \subset H^2(\Omega)$ satisfies $V \cap \mathcal{P}_1 = \emptyset$ and $0 < \nu < 1$, then (13) has a unique solution.*

When integrating by parts, all terms multiplied by $1-\nu$ cancel, as they yield cross derivative u_{xxyy}.

Thus corresponding PDE is

$$\Delta^2 u = f,$$

independent of Poisson's ration ν.

But natural boundary conditions do depend on ν.

Two essential boundary conditions are of physical interest.

One has natural boundary condition that depends on ν.

The "clamped" plate model consists of choosing $V^{\mathrm{c}} = \mathring{H}^2(\Omega)$, the subset of $H^2(\Omega)$ consisting of functions which vanish to second order on $\partial\Omega$:

$$V^{\mathrm{c}} = \left\{ v \in H^2(\Omega) \;:\; v = \frac{\partial v}{\partial n} = 0 \text{ on } \partial\Omega \right\}.$$

Rotation of plate also prescribed at boundary.

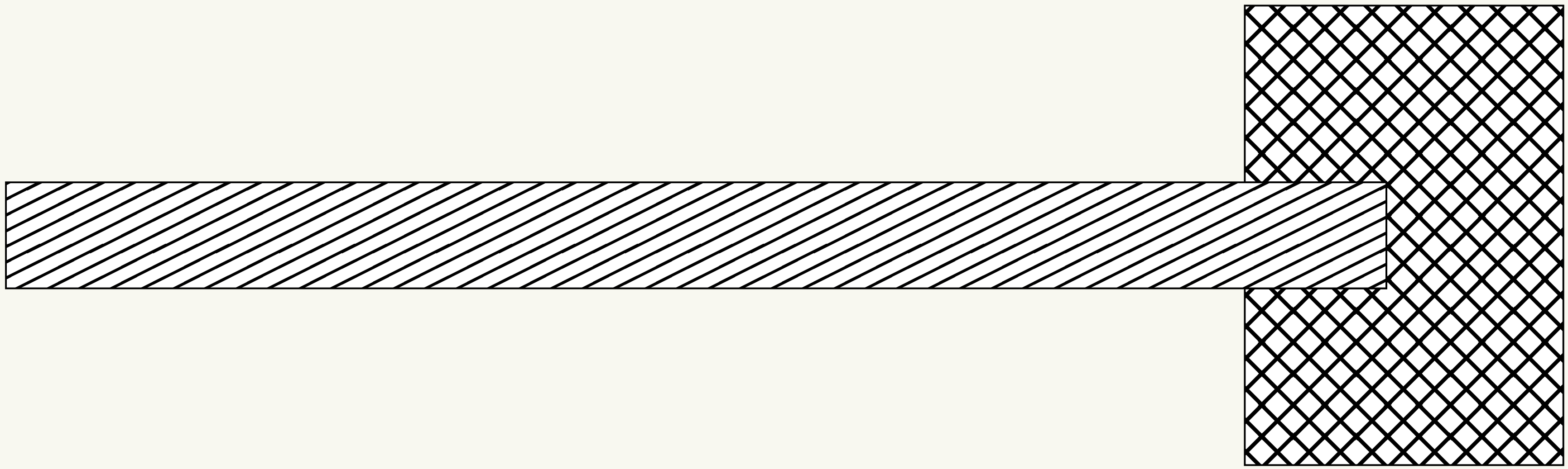


Figure 7: Plate clamped on right and free on left.

Simply-supported plate

Let V^{ss} denote the subset of $H^2(\Omega)$ consisting of functions which vanish (to first-order only) on $\partial\Omega$, i.e.,

$$V^{\mathrm{ss}} = \left\{ v \in H^2(\Omega) \ : \ v = 0 \text{ on } \partial\Omega \right\}.$$

Resulting model is “simply-supported” plate model.

Displacement, u, is held fixed (at a height of zero), yet the plate is free to rotate at the boundary.

Figure 8: Simply supported plate.

Natural boundary condition

In simply-supported case ($V = V^{\mathrm{ss}}$), there is another, *natural* boundary condition that holds.

Mixture of essential and natural boundary conditions hold on same part of $\partial\Omega$.

Natural boundary condition found using integration by parts, with v having nonzero normal derivative on $\partial\Omega$.

Then the “bending moment”

$$\Delta u + (1 - \nu) u_{tt}$$

must vanish on $\partial\Omega$ [Bergman and Schiffer 1953]

$u_{tt} =$ second derivative in tangential direction.

Theorem 0.2 *Suppose that V is any closed subspace satisfying $\mathring{H}^2(\Omega) \subset V \subset H^2(\Omega)$. If $f \in L^2(\Omega)$, and if $u \in H^4(\Omega)$ satisfies (13) with $F(v) = (f, v)$, then u satisfies*

$$\Delta^2 u = f$$

in the $L^2(\Omega)$ sense. For $V = V^{\mathrm{c}}$, u satisfies

$$u = \frac{\partial u}{\partial n} = 0 \text{ on } \partial\Omega$$

and for $V = V^{\mathrm{ss}}$, u satisfies

$$u = \Delta u + (1 - \nu) u_{tt} = 0 \text{ on } \partial\Omega.$$

To approximate (13), we need a subspace V_h of $H^2(\Omega)$.

For example, we could take a space based on the Argyris elements [5].

With either choice of V as above, if we choose V_h to satisfy the corresponding boundary conditions, we obtain the following.

Theorem 0.3 *If $V_h \subset V$ is based on Argyris elements of order $k \geq 5$ then there is a unique $u_h \in V_h$ such that*

$$a_P(u_h, v) = F(v) \quad \forall v \in V_h \text{ satisfying}$$

$$\begin{aligned} \|u - u_h\|_{H^2(\Omega)} &\leq C \inf_{v \in V_h} \|u - v\|_{H^2(\Omega)} \\ &\leq Ch^{k-1} \|u\|_{H^{k+1}(\Omega)}. \end{aligned} \tag{14}$$

More on plate-bending

For more details regarding the biharmonic equation model for plate bending, see the survey [18].

Several mixed methods reducing the biharmonic problem to a system of second-order problems have been developed [9, 11, 10].

The Babuška Paradox relates to the limit of polygonal approximations to a smooth boundary.

For example, let Ω be the unit disc, and let Ω_n denote regular polygons inscribed in Ω with n sides.

Then the Paradox [12, Chapter 18, Volume II] is that

solutions w_n of simply supported plate problems on Ω_n

converge to solution w of clamped plate problem on Ω as $n \to \infty$

The reason for the paradox is that, at each vertex of Ω_n, the gradient of v must vanish for any sufficiently smooth function v that vanishes on $\partial\Omega_n$.

This is illustrated in Figure 9.

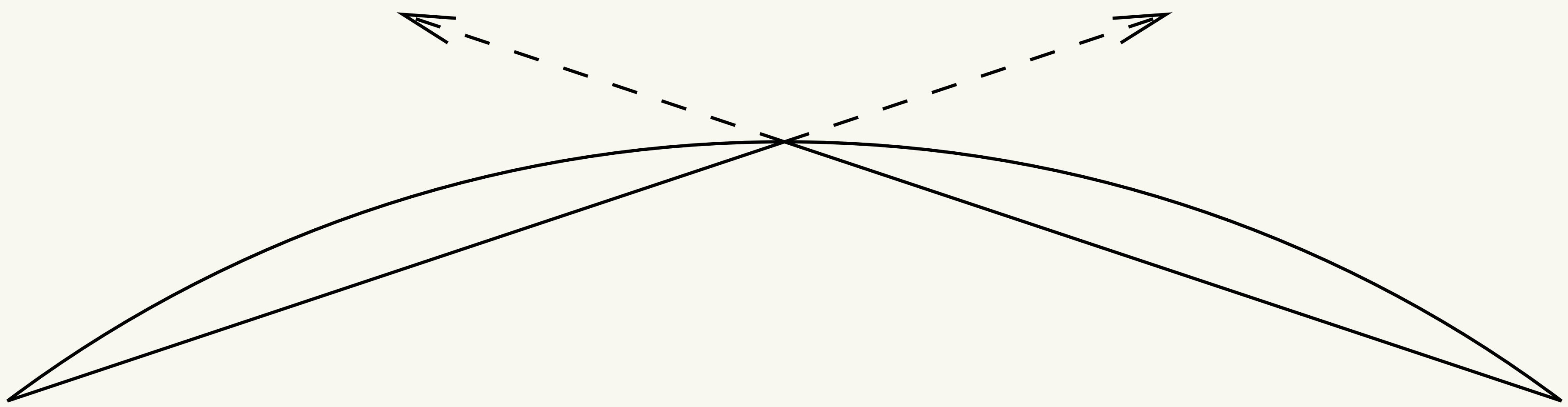


Figure 9: Polygonal approximation in the Babuška Paradox. At a vertex, the gradient of the simply supported solution must vanish as its tangential derivatives are zero in two independent directions.

Limits

In particular, ∇w_n must vanish at all vertices of the polygon Ω_n.

Thus in the limit, $\nabla w = 0$ at all points on the boundary, where $w = \lim_{n\to\infty} w_n$ and w_n denotes the solution of simply supported plate problem on Ω_n.

A corollary of this paradox is the following.

Discrete Babuška paradox

Suppose we approximate a smooth domain Ω by polygons Ω_n and form finite element approximation $w_{n,h}$ of simply supported plate problem with $h = 1/n$.

Then as $n \to \infty$ (equivalently, $h \to 0$), we expect that $w_{n,h}$ will converge to the solution w of the clamped plate problem on Ω, not the simply supported problem.

This numerical error is most insidious possible, in that convergence is likely to be quite stable.

No red flags to indicate that something is wrong.

Babuška Paradox history

Babuška Paradox widely studied [2, 14, 15, 16].

Now known as Babuška-Sapondzhyan Paradox [7, 13, 17].

Since the biharmonic equation arises in other contexts, including the Stokes equations, this paradox is of broader interest [20, 19, 8, 6].

Membranes are thin elastic media that do not resist bending.

Their models are similar in form to that of anti-plane strain, but for different reasons.

Membranes are similar to plates in that they are thin, but only the vertical deformation plays a role.

Thus we assume that

$$\nabla \mathbf{u} \approx \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ w_{,1} & w_{,2} & 0 \end{pmatrix} \text{ and } \boldsymbol{\epsilon} = \frac{1}{2} \begin{pmatrix} 0 & 0 & w_{,1} \\ 0 & 0 & w_{,2} \\ w_{,1} & w_{,2} & 0 \end{pmatrix}.$$

Therefore

$$\mathbf{T} = \mu \begin{pmatrix} 0 & 0 & w_{,1} \\ 0 & 0 & w_{,2} \\ w_{,1} & w_{,2} & 0 \end{pmatrix}.$$

and so

$$\nabla \cdot \mathbf{T} = \mu \begin{pmatrix} 0 \\ 0 \\ w_{,11} + w_{,22} \end{pmatrix} = \mu \begin{pmatrix} 0 \\ 0 \\ \Delta w \end{pmatrix}.$$

Thus the membrane problem reduces to the familiar Laplace equation,

$$-\mu \Delta w = f.$$

Locking is a numerical defect that can occur when Poisson's ratio $\nu \to \frac{1}{2}$ [3, 5].

The variational form for elasticity

$$\lambda \int_\Omega (\nabla \cdot \mathbf{u})(\nabla \cdot \mathbf{v})\, d\mathbf{x} + \mu \int_\Omega \left(\nabla \mathbf{u} + \nabla \mathbf{u}^t\right) : \nabla \mathbf{v}\, d\mathbf{x},$$

corresponds to iterated penalty method for Stokes with

$$\rho = \frac{2\lambda}{\mu} = \frac{4\nu}{1 - 2\nu} = \frac{2\nu}{\frac{1}{2} - \nu},$$

using physical constants conversions in Table 1.

Thus $\rho \to \infty$ when $\nu \to \frac{1}{2}$.

system	$K =$	$\mu =$	$\lambda =$	$\mu =$	$E =$	$\nu =$
K, μ	K	μ	$K - \frac{2}{3}\mu$	μ	$\frac{9K\mu}{3K+\mu}$	$\frac{3K-2\mu}{2(3K+\mu)}$
λ, μ	$\lambda + \frac{2}{3}\mu$	μ	λ	μ	$\mu\frac{3\lambda+2\mu}{\lambda+\mu}$	$\frac{\lambda}{2(\lambda+\mu)}$
E, ν	$\frac{E}{3(1-2\nu)}$	$\frac{E}{2(1+\nu)}$	$\frac{E\nu}{(1+\nu)(1-2\nu)}$	$\frac{E}{2(1+\nu)}$	E	ν

Table 1: Conversion guide for elasticity constants. E is Young's modulus. The shear modulus μ is sometimes denoted by G. Note that $\lambda \to \infty$ and $K \to \infty$ as $\nu \to \frac{1}{2}$.

It follows from the Stokes results that, if

$$\kappa = \min_{0 \neq v \in V_h,\, v \perp_a Z_h} \frac{(\mathcal{D}v, \mathcal{D}v)_\Pi}{a(v,v)} > 0, \tag{15}$$

then $\mathbf{u}_h^\rho \to \mathbf{u}_h^\infty$ as $\rho \to \infty$ ($\nu \to \frac{1}{2}$).

Of course, κ can depend on the mesh size h, so we write κ_h to indicate this.

More precisely, it follows that

$$\|\mathbf{u}_h^{\rho} - \mathbf{u}_h^{\infty}\|_a \leq \frac{1}{1+\rho\kappa_h}\|\mathbf{u}_h^{\infty}\|_a.$$

Here $\mathbf{u}_h^{\infty} \in V_h$ satisfies $\nabla\cdot \mathbf{u}_h^{\infty} = 0$ and

$$\tfrac{1}{2}\mu a_{\epsilon}(\mathbf{u}_h^{\infty}, \mathbf{v}) = F(\mathbf{v}) \text{ for all } \mathbf{v} \in V_h,$$

which we recognize as an approximation to the solution of a Stokes system.

Elasticity

Thus locking does **not** occur if the constant κ_h in (15) satisfies

$$\kappa_h > \kappa_0, \tag{16}$$

where $\kappa_0 > 0$ is independent of h.

For example, we know that (16) holds for Lagrange elements of sufficiently high degree, as discussed in Stokes chapter.

We leave as an exercise the exploration of the locking phenomenon.

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