Solving PDE’s with FEniCS

Scalar elliptic problems and mixed methods

Chapters 17–18

Introduction to

Automated Modeling

with FEniCS

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General scalar elliptic problem

The general scalar elliptic problem in divergence form

$$-\sum_{i,j=1}^{d} \frac{\partial}{\partial x_j}\left(\alpha_{ij}(\mathbf{x})\frac{\partial u}{\partial x_i}(\mathbf{x})\right) = f(\mathbf{x}) \qquad (1)$$

where the α_{ij} are given functions.

Posed with suitable boundary conditions of type considered previously:

$$u = 0 \text{ on } \Gamma, \qquad \frac{\partial u}{\partial n} = 0 \text{ on } \partial\Omega\backslash\Gamma,$$

Or Robin, pure Neumann, and so forth.

Ellipticity

To be elliptic, functions $\alpha_{ij}(\mathbf{x})$ must form a postive definite (often symmetric) matrix at almost every point $\mathbf{x}$:

$$C^{-1} \leq |\xi|^{-2} \sum_{i,j=1}^{d} \alpha_{ij}(\mathbf{x})\, \xi_i\, \xi_j \leq C \tag{2}$$

for all $0 \neq \xi \in \mathbb{R}^d$ and “for almost all” $\mathbf{x} \in \Omega$.

- Condition ignored on sets of measure zero.
- E. g., a lower-dimensional subset of Ω.

No need for the $\alpha_{ij}{}'s$ to be continuous.

In many physical applications they are not continuous.

Variational formulation

Interpretation of problem in classical terms difficult when α_{ij}'s are not differentiable.

But variational formulation quite simple. Define

$$a_\alpha(u,v) := \int_\Omega \sum_{i,j=1}^{d} \alpha_{ij}(\mathbf{x}) \frac{\partial u}{\partial x_i}(\mathbf{x}) \frac{\partial v}{\partial x_j}\, d\mathbf{x}. \tag{3}$$

Define $V = \left\{ v \in H^1(\Omega) \;:\; v = 0 \text{ on } \Gamma \right\}$ and

Find $u \in V$ such that

$$a_\alpha(u,v) = F(v) \quad \forall v \in V.$$

Discontinuous coefficients

Frequently coefficients in physical models vary so dramatically that it is appropriate to model them as discontinuous.

These often arise due to a change in materials or material properties.

Examples can be found in the modeling of nuclear reactors, porous media, semi-conductors, proteins in a solvent, and on and on.

But lack of continuity of coefficients has minimal effect.

Solution regularity

There is a subtle dependence of the regularity of the solution in the case of discontinuous coefficients [9].

It is not in general the case that the gradient of the solution is bounded.

However, from the variational derivation, we see that the gradient of the solution is always square integrable.

More is true: p-th power of the solution is integrable for $2 \leq p \leq P_C$.

$P_C > 2$ depends only on the ellipticity constant C.

Coercivity

Assumptions (2) imply coercivity and continuity.

For each $\mathbf{x} \in \Omega$, we take $\xi_i = v_{,i}(\mathbf{x})$ and apply (2):

$$\begin{aligned} C^{-1} \int_\Omega |\nabla \mathbf{v}(\mathbf{x})|^2 \, d\mathbf{x} &= C^{-1} \int_\Omega \sum_{i=1}^{d} v_{,i}(\mathbf{x})^2 \, d\mathbf{x} \\ &\leq \int_\Omega \sum_{i,j=1}^{d} \alpha_{ij}(\mathbf{x}) v_{,i}(\mathbf{x}) v_{,j}(\mathbf{x}) \, d\mathbf{x} = a_\alpha(v, v). \end{aligned} \tag{4}$$

With appropriate Dirichlet boundary conditions, this implies coercivity.

Continuity

Similarly, (2) implies that the bilinear form (3) is bounded:

$$
\begin{aligned}
a_\alpha(u,v) &= \int_\Omega \sum_{i,j=1}^{d} \alpha_{ij}(\mathbf{x}) u_{,i}(\mathbf{x}) v_{,j}(\mathbf{x})\, d\mathbf{x} \\
&\le C \int_\Omega |\nabla u(\mathbf{x})|\, |\nabla v(\mathbf{x})|\, d\mathbf{x} \\
&\le C \|u\|_{H^1(\Omega)} \|v\|_{H^1(\Omega)},
\end{aligned} \tag{5}
$$

using the Cauchy-Schwarz inequality.

Flux continuity

Using the variational form (3) of the equation (1), it is easy to see that the *flux*

$$\sum_{i=1}^{d} \alpha_{ij}(\mathbf{x})\frac{\partial u}{\partial x_i}(\mathbf{x})n_j \tag{6}$$

is continuous across an interface normal to $\mathbf{n}$ even when the α_{ij}'s are discontinuous across the interface.

This implies that the normal slope of the solution must have a jump (that is, the graph has a kink).

The derivation of (6) is just integration by parts, as we now show.

Example with a kink

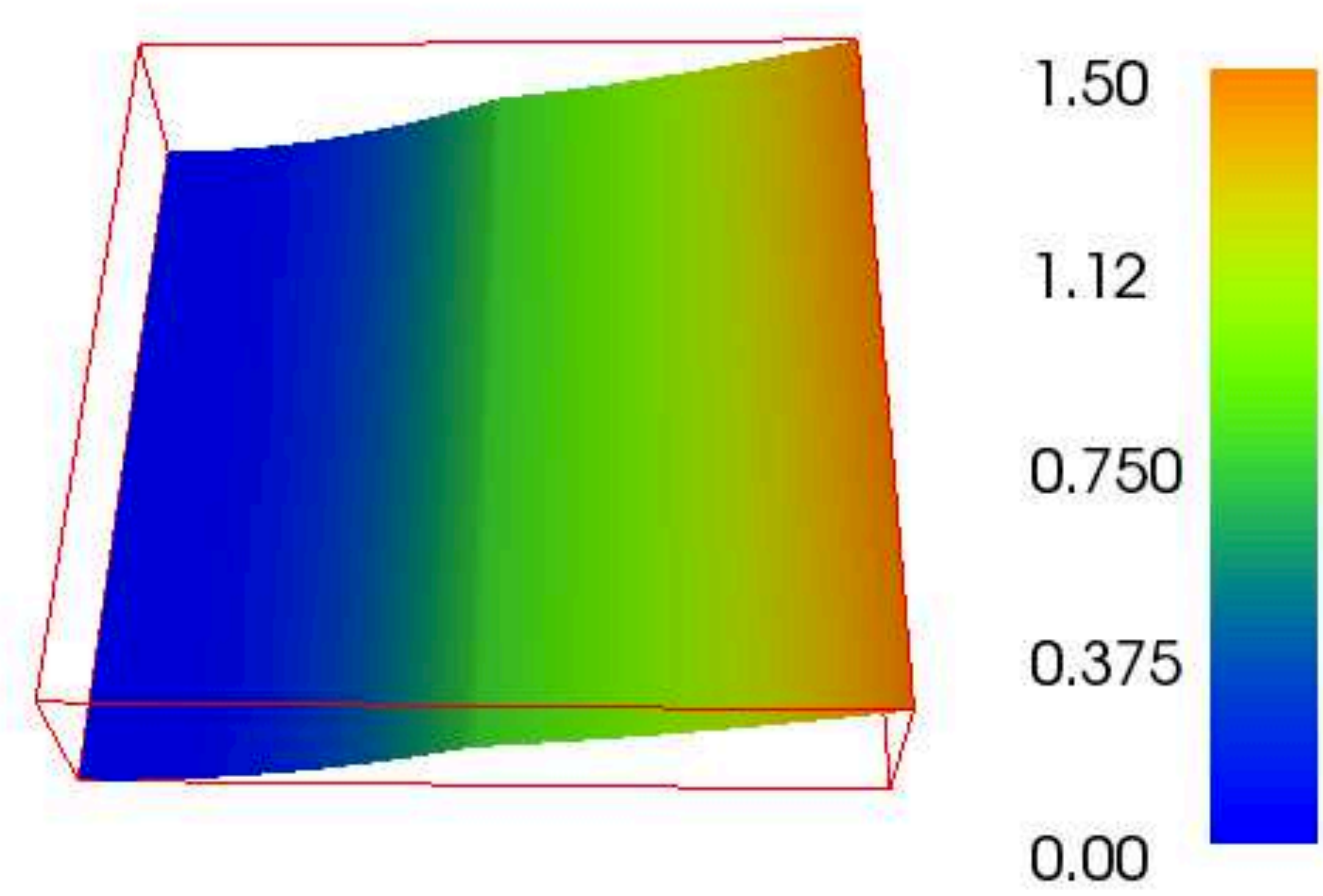


Figure 1: Scalar elliptic problem with “discontinuous” coefficient. Computed using piecewise linears on a 128×128 mesh and $\epsilon = 10^{-5}$.

Flux derivation

Suppose that $\Omega = \Omega_1 \cup \Omega_2$ and coefficients are smooth on each Ω_i, $i = 1, 2$, but jump across $B = \overline{\Omega_1} \cap \overline{\Omega_2}$.

For simplicity, suppose that $v = 0$ on $\partial\Omega$.

Define $\mathbf{w} = v\boldsymbol{\alpha}\nabla u$.

Apply the divergence theorem on each Ω_i separately to get

$$\begin{aligned} \oint_B v\mathbf{n}_i \cdot \boldsymbol{\alpha}\nabla u \, ds &= \int_{\Omega_i} \nabla\cdot \mathbf{w} \, dx \\ &= \int_{\Omega_i} (\boldsymbol{\alpha}\nabla u) \cdot \nabla v \, dx + \int_{\Omega_i} v\nabla\cdot (\boldsymbol{\alpha}\nabla u) \, dx \end{aligned} \tag{7}$$

Flux derivation

Summing this over i and using (1) we get

$$\oint_B v[\mathbf{n} \cdot \boldsymbol{\alpha} \nabla u]_B \, ds = a(u,v) - \int_\Omega f v \, d\mathbf{x} = 0, \tag{8}$$

where the jump expression $[\phi]_B$ is defined by

$$[\phi(x)]_B = \lim_{h \to 0} \phi(x - h\mathbf{n}) - \lim_{h \to 0} \phi(x + h\mathbf{n})$$

and $\mathbf{n}$ is either $\mathbf{n}_1$ or $\mathbf{n}_2 = -\mathbf{n}_1$.

$[\mathbf{n} \cdot \boldsymbol{\alpha} \nabla u]_B$ is the same whether $\mathbf{n} = \mathbf{n}_1$ or $\mathbf{n} = \mathbf{n}_2$.

$[\mathbf{n} \cdot \boldsymbol{\alpha} \nabla u]_B$ is the jump in the flux (6) across B.

Since

$$\oint_B v[\mathbf{n} \cdot \boldsymbol{\alpha} \nabla u]_B \, ds = 0$$

holds for all v vanishing on $\partial\Omega$, we conclude that

$$[\mathbf{n} \cdot \boldsymbol{\alpha} \nabla u]_B = 0$$

everywhere on B.

This completes the proof of flux continuity.

Piecewise constant example

Take $\boldsymbol{\alpha}$ to be a scalar function times the identity matrix:

$$\alpha_{ij}(\mathbf{x}) = \delta_{ij}\alpha(\mathbf{x}),$$

where δ_{ij} is the Kronecker delta and

$$(1/C) \geq \alpha(\mathbf{x}) \geq C > 0$$

for all $\mathbf{x} \in \Omega$. Let $\Omega = [0,1]^2$, with

$$\Omega_1 = [0, 1/2] \times [0, 1] \quad \text{and} \quad \Omega_2 = [1/2, 1] \times [0, 1].$$

Thus

$$B = \left\{(x, y) \; : \; x = \tfrac{1}{2}\right\}.$$

Piecewise constant bilinear form

Define

$$a(u,v) = \int_\Omega \alpha \nabla u \cdot \nabla v \, d\mathbf{x},$$

where we take

$$\alpha(x,y) = \begin{cases} 1 & x < 1/2 \\ 3 & x > 1/2. \end{cases} \tag{9}$$

Consider the problem (1), posed variationally, with $f \equiv -6$ and with Dirichlet boundary conditions $u = 0$ on $\{(0,y) \ : \ y \in [0,1]\}$ and $u = 3/2$ on $\{(1,y) \ : \ y \in [0,1]\}$.

Thus the variational space is

$$V = \left\{ v \in H^1(\Omega) \ : \ v(x,y) = 0 \text{ if } x = 0 \text{ or } x = 1, \forall y \in [0,1] \right\}.$$

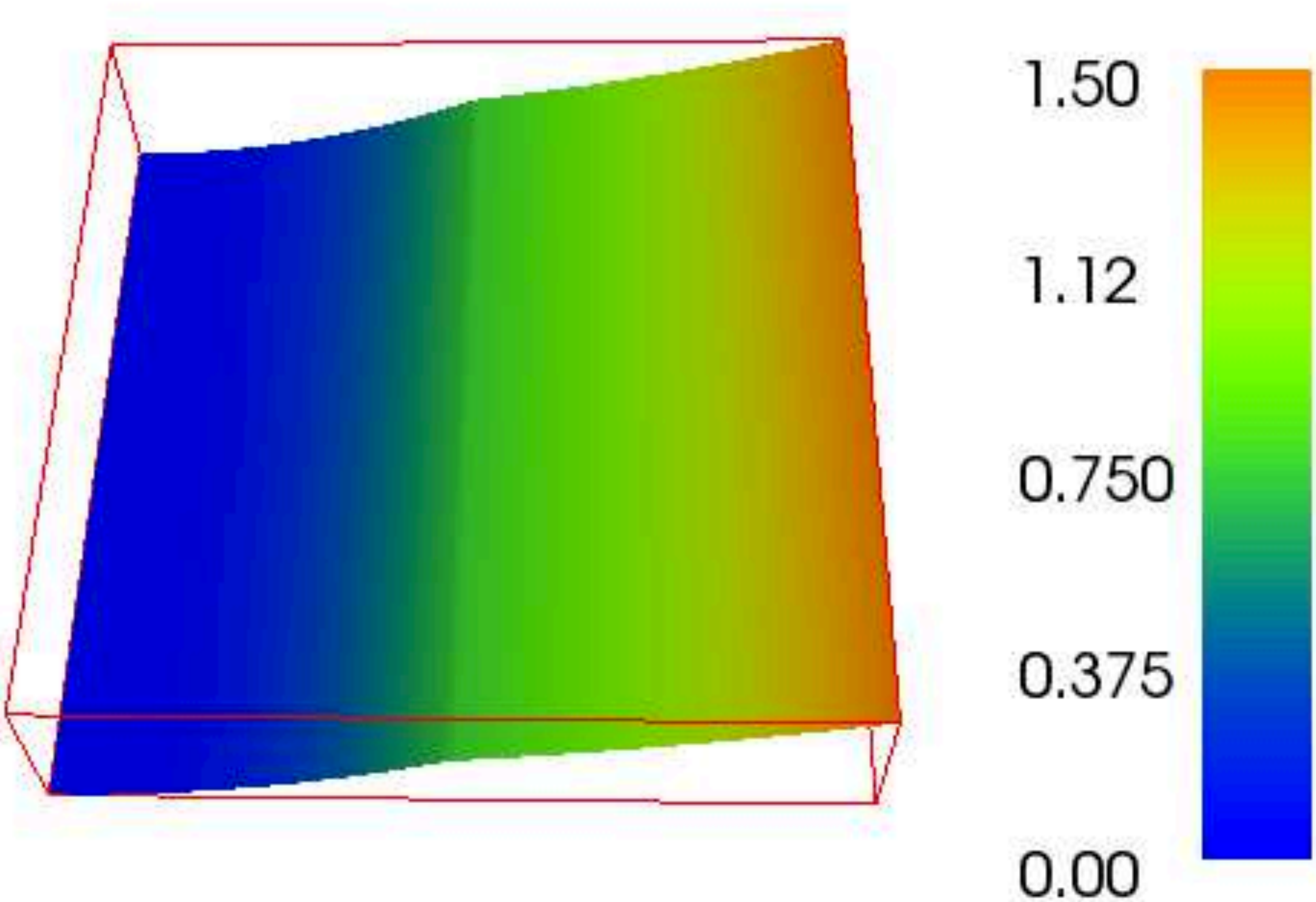
The exact solution u satisfies

$$u(x,y) = \begin{cases} 3x^2 & x < 1/2 \\ \frac{1}{2} + x^2 & x > 1/2. \end{cases} \tag{10}$$

Note that $\alpha \frac{\partial u}{\partial x} = 6$ for all $x \neq \frac{1}{2}$.

The solution is depicted in Figure 1, and the “kink” in the solution across B is evident.

Example with a kink



Piecewise constant example

To render the computational problem simpler, we defined $\alpha(x) = 2 + \tanh(M(x - 0.5))$.

The computation represented in Figure 1 was done with $M = 10^5$.

This definition of α is consistent with many applications where coefficient is smooth but changes abruptly over a length scale of $\mathcal{O}(M^{-1})$.

It is significant that the computations do not depend significantly on M.

Discontinuous coefficients appear in a model for dielectric behavior of protein in water of the form

$$-\nabla \cdot (\mathcal{E} \nabla u) = \sum_{i=1}^{N} c_i \, \delta_{\mathbf{x}_i} \quad \text{in } \mathbb{R}^3$$
$$u(\mathbf{x}) \to 0 \quad \text{as } \mathbf{x} \to \infty, \tag{11}$$

Dielectric constant $\mathcal{E}$ small inside the protein (domain Ω) and large outside.

Point charges at $\mathbf{x}_i$ modeled via Dirac δ-functions $\delta_{\mathbf{x}_i}$.

Constants c_i corresponds to the charge at $\mathbf{x}_i$.

Confusion was caused about error estimators due to the need for resolving point singularities [8].

Limited use of error estimators for such models.

Subsequently [5] shown that a splitting improves efficacy of error estimators.

Error estimators necessarily indicate large errors anywhere there are fixed charges, thus throughout the protein, not primarily at the interface.

Singularity due to point charges is more severe than that caused by jump in dielectric coefficient $\mathcal{E}$.

Consider a splitting u=v+w where

$$v(\mathbf{x}) = \sum_{i=1}^{N} \frac{c_i}{|\mathbf{x} - \mathbf{x}_i|}. \tag{12}$$

Assume units chosen so that fundamental solution of $-\mathcal{E}_0 \Delta u = \delta_0$ is $1/|\mathbf{x}|$, where $\mathcal{E}_0$ is dielectric constant in Ω.

By definition, w is harmonic in both Ω and $\mathbb{R}^3 \backslash \Omega$, and $w(\mathbf{x}) \to 0$ as $\mathbf{x} \to \infty$.

But the jump in the normal derivative of w across the interface $B = \partial\Omega$ is not zero.

Equation for w

Define

$$\left[\mathcal{E}\frac{\partial w}{\partial n}\right]_B = \mathcal{E}_0\frac{\partial w}{\partial n}\Big|_{B-} - \mathcal{E}_\infty\frac{\partial w}{\partial n}\Big|_{B+},$$

where

- $B-$ denotes the inside of the interface,
- $B+$ denotes the outside of the interface,
- and $\mathbf{n}$ denotes the outward normal to Ω.

The solution u of (11) satisfies $\left[\mathcal{E}\frac{\partial u}{\partial n}\right]_B = 0$, so

$$\left[\mathcal{E}\frac{\partial w}{\partial n}\right]_B = (\mathcal{E}_\infty - \mathcal{E}_0)\frac{\partial v}{\partial n}\Big|_B.$$

Splitting oundary condition

Intergrating by parts, we have

$$a(w,\phi) = \oint_B \left[\mathcal{E}\frac{\partial w}{\partial n}\right]_B \phi\, ds = (\mathcal{E}_\infty - \mathcal{E}_0) \oint_B \frac{\partial v}{\partial n}\phi\, ds$$

for all test functions ϕ.

The linear functional F defined by

$$F(\phi) = (\mathcal{E}_\infty - \mathcal{E}_0) \oint_B \frac{\partial v}{\partial n}\phi\, ds \tag{13}$$

is well defined for any test function, since v is smooth except at singular points x_i, which we assume are in interior of Ω, not on boundary $B = \partial\Omega$.

Thus w is defined by standard variational formulation but need to truncate the infinite domain

For example, we can define

$$B_R = \left\{ \mathbf{x} \in \mathbb{R}^3 \;:\; |\mathbf{x}| < R \right\},$$

and define

$$a_R(\phi, \psi) = \int_{B_R} \mathcal{E} \nabla \phi \cdot \nabla \psi \, d\mathbf{x},$$

and solve for $w_R \in H_0^1(B_R)$ such that

$$a_R(w_R, \psi) = F(\psi) \quad \forall \psi \in H_0^1(B_R), \tag{14}$$

where F is defined in (13). Then $w_R \to w$ as $R \to \infty$.

Consider a single point charge at the origin of a spherical domain of radius $\rho > 0$:

$$\Omega = \left\{ \mathbf{x} \in \mathbb{R}^3 \ : \ |\mathbf{x}| < \rho \right\}$$

Let $\mathcal{E}_0$ denote the dielectric constant in Ω

and $\mathcal{E}_\infty$ denote the dielectric constant in $\mathbb{R}^3 \backslash \Omega$.

Then the solution to (11) is

$$u(\mathbf{x}) = \begin{cases} \frac{1}{|\mathbf{x}|} - \frac{c}{\rho} & |\mathbf{x}| \leq \rho \\ \frac{1-c}{|\mathbf{x}|} & |\mathbf{x}| \geq \rho, \end{cases} \tag{15}$$

where $c = 1 - \mathcal{E}_0 / \mathcal{E}_\infty$.

Point-charge verification

The verification is as follows.

In Ω, we have $\Delta u = \delta_{\mathbf{0}}$.

In $\mathbb{R}^3 \backslash \Omega$, we have $\Delta u = 0$.

At the interface $B = \partial\Omega = \left\{ \mathbf{x} \in \mathbb{R}^3 \ : \ |\mathbf{x}| = \rho \right\}$,

$$\frac{\partial u}{\partial n}\Big|_{B-} = \frac{\partial u}{\partial r}(\rho-) = \frac{-1}{\rho^2}, \quad \frac{\partial u}{\partial n}\Big|_{B+} = \frac{\partial u}{\partial r}(\rho+) = \frac{-(1-c)}{\rho^2},$$

where

$B-$ denotes the inside of the interface and

$B+$ denotes the outside of the interface.

Model problem

Thus the jump is given by

$$\left[\mathcal{E}\frac{\partial u}{\partial n}\right]_B = \mathcal{E}_0\frac{\partial u}{\partial n}\Big|_{B-} - \mathcal{E}_\infty\frac{\partial u}{\partial n}\Big|_{B+} = \frac{-\mathcal{E}_0 + (1-c)\mathcal{E}_\infty}{\rho^2} = 0.$$

In this case, $v(\mathbf{x}) = 1/|\mathbf{x}|$, so

$$w(\mathbf{x}) = -c\begin{cases} \frac{1}{\rho} & |\mathbf{x}| \le \rho \\ \frac{1}{|\mathbf{x}|} & |\mathbf{x}| \ge \rho. \end{cases} \tag{16}$$

Thus if we solve numerically for w, we have a much smoother problem.

But as $\rho \to 0$, w becomes more singular.

Mixed formulations

Name “mixed method” applied to a variety of finite element methods having more than one approximation space.

Typically one or more of the spaces play the role of Lagrange multipliers which enforce constraints.

Name and many concepts originated in solid mechanics [1] where desirable to have more accurate approximation of derivatives of the displacement.

But for the Stokes equations for viscous fluid flow, the natural Galerkin approximation is a mixed method.

Mixed-method features

Mixed methods have features that make them attractive.

- For problems like (1), emphasis switches from approximating solution to approximating its gradient.
- Role of essential and natural boundary conditions is reversed.

With mixed methods for scalar elliptic problems (1),

- the Neumann condition becomes essential, whereas
- the Dirichlet condition is imposed only weakly through the variational equation.

But not all choices of finite element spaces converge.

Approximability alone insufficient to guarantee success.

Coercivity

We will focus on mixed methods in which there are two bilinear forms and two approximation spaces.

There are two key conditions that lead to the success of a mixed method.

Both are in some sense coercivity conditions for the bilinear forms.

One of these will look like a standard coercivity condition, while the other, often called the *inf-sup* condition, takes a new form.

To motivate the mixed method, we take a particular application in which the mixed method arises naturally.

A simplified model [6] for miscible displacement of a fluid in a porous medium, occupying a domain Ω, takes the form

$$-\sum_{i,j=1}^{d} \frac{\partial}{\partial x_i}\Big(\alpha_{ij}(\mathbf{x})\frac{\partial p}{\partial x_j}(\mathbf{x})\Big) = f(\mathbf{x}) \text{ in } \Omega, \tag{17}$$

where p is the pressure (we take an inhomogeneous right-hand-side for generality).

Darcy's Law postulates that the fluid velocity $\mathbf{u}$ is related to the gradient of p by

$$u_i(\mathbf{x}) = \sum_{j=1}^{d} \alpha_{ij}(\mathbf{x}) \frac{\partial p}{\partial x_j}(\mathbf{x}) \quad \forall i = 1, \ldots, d. \tag{18}$$

Using matrix and vector notation, $\mathbf{u} = \boldsymbol{\alpha} \nabla p$.

Coefficients α_{ij}, assumed to form a symmetric, positive-definite matrix $\boldsymbol{\alpha}$ (almost everywhere) are related to the porosity of the medium

Frequently coefficients are discontinuous where materials change.

Combining (17) and Darcy's Law (18), we find

$$-\nabla\cdot \mathbf{u} = f \quad \text{in } \Omega.$$

Variational formulation for (17) derived by letting

$\mathbf{A}(\mathbf{x}) =$ inverse of the coefficient matrix $\boldsymbol{\alpha} = (\alpha_{ij})$

and by writing $\nabla p = \mathbf{A}\mathbf{u}$.

Define

$$a(\mathbf{u}, \mathbf{v}) := \sum_{i,j=1}^{d} \int_{\Omega} A_{ij}(\mathbf{x}) u_i(\mathbf{x}) v_j(\mathbf{x})\, d\mathbf{x}. \tag{19}$$

Mixed derivation

Multiply (dot product) equation

$$\nabla p = \mathbf{A}\mathbf{u}$$

by $\mathbf{v}$, and integrate over Ω, to get

$$\begin{aligned}\int_\Omega \nabla p(\mathbf{x}) \cdot \mathbf{v}(\mathbf{x})\, d\mathbf{x} &= \int_\Omega \big(\mathbf{A}(\mathbf{x})\mathbf{u}(\mathbf{x})\big) \cdot \mathbf{v}(\mathbf{x})\, d\mathbf{x} \\ &= a(\mathbf{u}, \mathbf{v}).\end{aligned} \tag{20}$$

Define

$$b(\mathbf{w}, q) = \int_\Omega \big(\nabla\cdot \mathbf{w}(\mathbf{x})\big)\, q(\mathbf{x})\, d\mathbf{x}. \tag{21}$$

Mixed derivation

From the divergence theorem:

$$\int_\Omega \nabla\cdot \big(\mathbf{w}(\mathbf{x})q(\mathbf{x})\big)\,d\mathbf{x} = \oint_{\partial\Omega} q(\mathbf{x})\,\mathbf{w}(\mathbf{x})\cdot\mathbf{n}(\mathbf{x})\,d\mathbf{x},$$

together with the product formula

$$\nabla\cdot \big(\mathbf{w}(\mathbf{x})q(\mathbf{x})\big) = \mathbf{w}(\mathbf{x})\cdot\nabla q(\mathbf{x}) + \nabla\cdot\mathbf{w}(\mathbf{x})\,q(\mathbf{x}),$$

and using $b(\mathbf{w},q) = \int_\Omega \big(\nabla\cdot\mathbf{w}(\mathbf{x})\big)\,q(\mathbf{x})\,d\mathbf{x}$, we get

$$\begin{aligned}\int_\Omega \mathbf{w}(\mathbf{x})\cdot\nabla q(\mathbf{x})\,d\mathbf{x} \\ = -b(\mathbf{w},q) + \oint_{\partial\Omega} q(\mathbf{x})\,\mathbf{w}(\mathbf{x})\cdot\mathbf{n}(\mathbf{x})\,d\mathbf{x}.\end{aligned} \tag{22}$$

Model problem

Combining (20) and (22), we get

$$a(\mathbf{u}, \mathbf{v}) + b(\mathbf{v}, p) = \oint_{\partial\Omega} p(\mathbf{x})\, \mathbf{v}(\mathbf{x}) \cdot \mathbf{n}(\mathbf{x})\, d\mathbf{x}.$$

Define a new space $\underset{\sim}{V}$ by

$$\underset{\sim}{V} := \left\{ \mathbf{v} \in L^2(\Omega)^d \ : \ \nabla\cdot \mathbf{v} \in L^2(\Omega),\ \mathbf{v} \cdot \mathbf{n} = 0 \text{ on } \partial\Omega \backslash \Gamma \right\}.$$

Also define $\Pi = L^2(\Omega)$.

Mixed formulation of (17)

Then the solution of (17) with the boundary conditions

$$\mathbf{u}\cdot\mathbf{n} = 0 \quad \text{on } \partial\Omega\backslash\Gamma \qquad \text{and} \qquad p = g \quad \text{on } \Gamma$$

satisfies $\mathbf{u} \in \underset{\sim}{V}$, $p \in \Pi$, and solves

$$\begin{aligned} a(\mathbf{u},\mathbf{v}) + b(\mathbf{v},p) &= \oint_\Gamma g(\mathbf{x})\,\mathbf{v}(\mathbf{x})\cdot\mathbf{n}(\mathbf{x})\,d\mathbf{x} \quad \forall \mathbf{v} \in \underset{\sim}{V}, \\ b(\mathbf{u},q) &= F(q) \quad \forall q \in \Pi\,, \end{aligned} \tag{23}$$

where $F(q) = -\int_\Omega f(\mathbf{x})\,q(\mathbf{x})\,d\mathbf{x}$.

Dirichlet condition $p = g$ on Γ appears as a natural boundary condition, imposed variationally.

(Essential boundary condition in a standard variational approach.)

The space $\underset{\sim}{V}$ is based on the space called

$$H_{\mathrm{div}}(\Omega) \text{ [10, Chapter 20, page 99]}$$

that has a natural norm given by

$$\|\mathbf{v}\|^2_{H_{\mathrm{div}}(\Omega)} = \|\mathbf{v}\|^2_{L^2(\Omega)^d} + \|\nabla\cdot\mathbf{v}\|^2_{L^2(\Omega)}\,; \tag{24}$$

$H_{\mathrm{div}}(\Omega)$ is a Hilbert space with inner-product given by

$$(\mathbf{u},\mathbf{v})_{H(\mathrm{div};\Omega)} = (\mathbf{u},\mathbf{v})_{L^2(\Omega)^d} + (\nabla\cdot\mathbf{u},\nabla\cdot\mathbf{v})_{L^2(\Omega)}.$$

The trace $\mathbf{v}\cdot\mathbf{n} = 0$ on $\partial\Omega$ is well defined for $\mathbf{v}\in H_{\mathrm{div}}(\Omega)$ [10], but the tangential derivatives of a general function $\mathbf{v}\in H_{\mathrm{div}}(\Omega)$ are not well defined.

Meaning of Dirichlet condition

The meaning of the boundary condition $p = g$ on Γ must be interpreted carefully.

If p is smooth enough, it will be defined pointwise.

But otherwise its meaning is like that of the

Neumann condition for the Laplace equation.

Will be enforced only weakly.

Coercivity

The bilinear form $a(\cdot, \cdot)$ is not coercive on all of $\underset{\sim}{V}$, but it is coercive on the subspace $\mathbf{Z}$ of divergence-zero functions, since on this subspace the inner-product $(\cdot, \cdot)_{H(\mathrm{div};\Omega)}$ is the same as the $L^2(\Omega)$ inner-product.

In particular, this proves uniqueness of solutions.

Suppose that F and g are zero.

Then $\mathbf{u} \in \mathbf{Z}$ and $a(\mathbf{u}, \mathbf{u}) = 0$.

Thus $\|\mathbf{u}\|_{L^2(\Omega)} = 0$, that is, $\mathbf{u} \equiv \mathbf{0}$.

To show that $p = 0$, we need to invoke the inf-sup condition.

Existence and stability follows from inf-sup condition.

Recall the space $\Pi_0 = \left\{ q \in L^2(\Omega) \ : \ \int_\Omega q(\mathbf{x})\, d\mathbf{x} = 0 \right\}$.

There is a constant C such that for all $q \in \Pi_0$

$$\begin{aligned} \|q\|_{L^2(\Omega)} &\leq C \sup_{0 \neq \mathbf{v} \in H_0^1(\Omega)} \frac{b(\mathbf{v}, q)}{\|\mathbf{v}\|_{H^1(\Omega)}} \\ &\leq C' \sup_{0 \neq \mathbf{v} \in \underset{\sim}{V}} \frac{b(\mathbf{v}, q)}{\|\mathbf{v}\|_{H(\mathrm{div};\Omega)}}, \end{aligned} \tag{25}$$

where first inequality is same as Stokes and second follows from the inclusion $H^1(\Omega)^d \subset H(\mathrm{div};\Omega)$ and the less restrictive boundary conditions on $\mathbf{v} \in \underset{\sim}{V}$.

Determining the constant

Means that inf-sup condition determines solution p in the mixed formulation (23) up to a constant.

Could not expect more, since solution of pure Neumann problem can be determined only up to a constant.

Mixed formulation of pure Neumann case has $\Gamma = \emptyset$.

Thus $\int_\Omega \nabla\cdot \mathbf{v}\, d\mathbf{x} = 0$ for all $\mathbf{v} \in \underset{\sim}{V}$.

For well posed problem, must have $\Gamma \neq \emptyset$.

This restriction is easy to motivate, as follows.

If $\underset{\sim}{V} = \{\mathbf{v} \in H_{\mathrm{div}}(\Omega) \ : \ \mathbf{v} \cdot \mathbf{n} = 0$ on $\partial\Omega\}$, then the divergence theorem implies that

$$\int_\Omega \nabla \cdot \mathbf{v}(\mathbf{x})\, d\mathbf{x} = 0 \quad \forall \mathbf{v} \in \underset{\sim}{V}.$$

Thus $b(\mathbf{v}, p) = 0$ for any constant p. Therefore

$$(\mathbf{u}, p) = (\mathbf{0}, \text{constant})$$

solves variational formulation (23) for $F = 0$ and $g = 0$.

Thus it is essential to have $\Gamma \neq \emptyset$.

When $\Gamma \neq \emptyset$, we can take any $\mathbf{w}$ such that

$$\mathbf{w} \cdot \mathbf{n} = 1 \text{ on } \Gamma$$

and by the divergence theorem we are assured that

$$\int_\Omega \nabla \cdot \mathbf{w}(\mathbf{x})\, d\mathbf{x} = |\Gamma|.$$

Let $\overline{q} = \frac{1}{|\Omega|} \int_\Omega q(\mathbf{x})\, d\mathbf{x}$. Then

$$\begin{aligned} \|q - \overline{q}\|_{L^2(\Omega)} &\leq C \sup_{0 \neq \mathbf{v} \in H_0^1(\Omega)} \frac{b(\mathbf{v}, q - \overline{q})}{\|\mathbf{v}\|_{H^1(\Omega)}} \\ &= C \sup_{0 \neq \mathbf{v} \in H_0^1(\Omega)} \frac{b(\mathbf{v}, q)}{\|\mathbf{v}\|_{H^1(\Omega)}} \end{aligned} \tag{26}$$

Define B by

$$B = \sup_{0 \neq \mathbf{v} \in \underset{\sim}{V}} \frac{b(\mathbf{v}, q)}{\|\mathbf{v}\|_{H^1(\Omega)}}$$

So we have

$$\|q - \overline{q}\|_{L^2(\Omega)} \leq CB.$$

Therefore

$$\|q\|_{L^2(\Omega)} \leq \|q - \overline{q}\|_{L^2(\Omega)} + \|\overline{q}\|_{L^2(\Omega)} \leq CB + \|\overline{q}\|_{L^2(\Omega)}.$$

Also

$$\begin{aligned} \|\overline{q}\|_{L^2(\Omega)} &= |\Omega|^{1/2}\,|\overline{q}| = |\Omega|^{1/2}|\Gamma|^{-1}|b(\mathbf{w}, \overline{q})| \\ &\leq |\Omega|^{1/2}|\Gamma|^{-1}\big|b(\mathbf{w}, q) - b(\mathbf{w}, q - \overline{q})\big|. \end{aligned} \tag{27}$$

inf-sup revisited

Let $c = |\Omega|^{1/2}|\Gamma|^{-1}$. Then

$$\begin{aligned} \|\overline{q}\|_{L^2(\Omega)} &\le c\big(|b(\mathbf{w},q)| + \|\nabla\cdot \mathbf{w}\|_{L^2(\Omega)}\|q-\overline{q}\|_{L^2(\Omega)}\big) \\ &\le c|b(\mathbf{w},q)| + c\|\nabla\cdot \mathbf{w}\|_{L^2(\Omega)}CB. \end{aligned} \tag{28}$$

Therefore

$$\|q\|_{L^2(\Omega)} \le CB\big(1 + c\|\nabla\cdot \mathbf{w}\|_{L^2(\Omega)}\big) + c|b(\mathbf{w},q)|. \tag{29}$$

Clearly

$$|b(\mathbf{w},q)| \le \|\mathbf{w}\|_{H^1(\Omega)}B$$

So

$$\|q\|_{L^2(\Omega)} \le C'B.$$

Discrete mixed formulation

Now let $\underset{\sim}{V}_h \subset \underset{\sim}{V}$ and $\Pi_h \subset \Pi$.

Consider variational problem to find $u_h \in \underset{\sim}{V}_h$ and $p_h \in \Pi_h$ such that

$$\begin{aligned} a(u_h, v) + b(v, p_h) &= F(v) \quad \forall v \in \underset{\sim}{V}_h\,, \\ b(u_h, q) &= 0 \quad \forall q \in \Pi_h\,. \end{aligned} \tag{30}$$

Case of inhomogeneous right-hand-side in second equation is considered in [2, Section 10.5].

Spaces to consider:

Taylor-Hood, Scott-Vogelius, BDM.

W_h^k denotes space of continuous piecewise polynomials of degree k (with no boundary conditions imposed).

Let the space $\underset{\sim}{V}_h$ be defined by

$$\underset{\sim}{V}_h = \left\{ \mathbf{v} \in W_h^k \times W_h^k \; : \; \mathbf{v} = 0 \text{ on } \partial\Omega \right\}. \tag{31}$$

and the space Π_h be defined by

$$\Pi_h = \left\{ q \in W_h^{k-1} \; : \; \int_\Omega q(\mathbf{x})\, d\mathbf{x} = 0 \right\}. \tag{32}$$

“converges but with a loss of convergence order and without convergence of divergence of velocities” [4].

For some computational experiments, see [7].

Taylor-Hood issues

Taylor-Hood satisfies the inf-sup condition on $H(\mathrm{div};\Omega)$

Proof is similar to proof for conintuous problem.

So the problem with Taylor-Hood must be lack of uniform coercivity.

To approximate the scalar elliptic problem (17) by a mixed method, have to contend with the fact that the corresponding form

$a(\cdot,\cdot)$ is not coercive on all of $\underset{\sim}{V}$.

Coercivity problem

$a(\cdot,\cdot)$ is clearly coercive on the space

$$\mathbf{Z} = \{\mathbf{v} \in H_{\mathrm{div}}(\Omega) \ : \ \nabla\cdot\mathbf{v} = 0\}$$

so that (23) is well-posed.

However, some care is required to assure that it is well-posed as well on

$$\mathbf{Z}_h = \left\{\mathbf{v} \in \underset{\sim}{V}_h \ : \ b(\mathbf{v}, q) = 0 \quad \forall q \in \Pi_h\right\}.$$

One simple solution is to insure that $\mathbf{Z}_h \subset \mathbf{Z}$, and there are many ways this can be done.

Choice of spaces: Scott-Vogelius

Let $\underset{\sim}{V}_h$ be as given in (31) and let $\Pi_h = \nabla\cdot \underset{\sim}{V}_h$.

Then (under certain mild restrictions on the mesh these spaces can be used.

The iterated penalty method can be used to solve the linear system using $\Pi_h = \mathcal{D}\underset{\sim}{V}_h$ without having explicit information about the structure of Π_h.

Another pair of spaces of interest is BDM [3] for $\underset{\sim}{V}_h$ and DG (discontinuous Galerkin) for Π_h.

More precisely, the inf-sup stable pair of spaces is BDM(k) for u and DG($k-1$) for p.

BDM and DG

The space DG(k) consists of discontinuous polynomials of degree k.

The BDM spaces are defined by

$$\text{BDM}(k) = \text{DG}(k) \cap H_{\text{div}}(\Omega).$$

The BDM spaces are the largest subset of DG(k) which are suitable for the mixed method.

The BDM spaces are more complicated to describe, so we limit our description to BDM(1).

BDM(1) description

The space BDM(1) consists of piecewise linear, vector-valued functions.

We require that BDM(1)$\subset H_{\mathrm{div}}(\Omega)$, so we must have the normal components of $\mathbf{v} \in$ BDM(1) continuous across edges.

Thus we can define BDM(1) as

> subset of vector-valued functions in DG(1)$\times$DG(1)
>
> with normal components continuous across edges.

Remains to determine what this means in terms of nodal parameters to represent functions locally.

BDM nodal parameters

On each triangle, a linear, vector-valued function has 6 degrees of freedom (3 for each component of the vector-valued function).

On the other hand, continuity of the normal components of a linear function requires two constraints per edge.

Thus we have 6 constraints and 6 degrees of freedom.

Of course, this alone does not mean that the corresponding system of equations is invertible, but this can be proved as follows.

BDM confirmation

Let $\mathbf{v}$ be a linear, vector-valued function such that $\mathbf{v} \cdot \mathbf{n} = 0$ on each edge of a triangle.

Then by divergence theorem, $\nabla \cdot \mathbf{v} = 0$ in the triangle.

Thus we can write $\mathbf{v} = \operatorname{curl} q$ for a quadratic (scalar) function q.

But since $\mathbf{v} \cdot \mathbf{n} = 0$ on each edge of the triangle, $\nabla q \cdot \mathbf{t} = 0$ on each edge where $\mathbf{t}$ is tangent.

Thus q is constant on each edge, and thus it must be constant on the entire triangle.

Thus we conclude that $\mathbf{v} = 0$.

Test solution

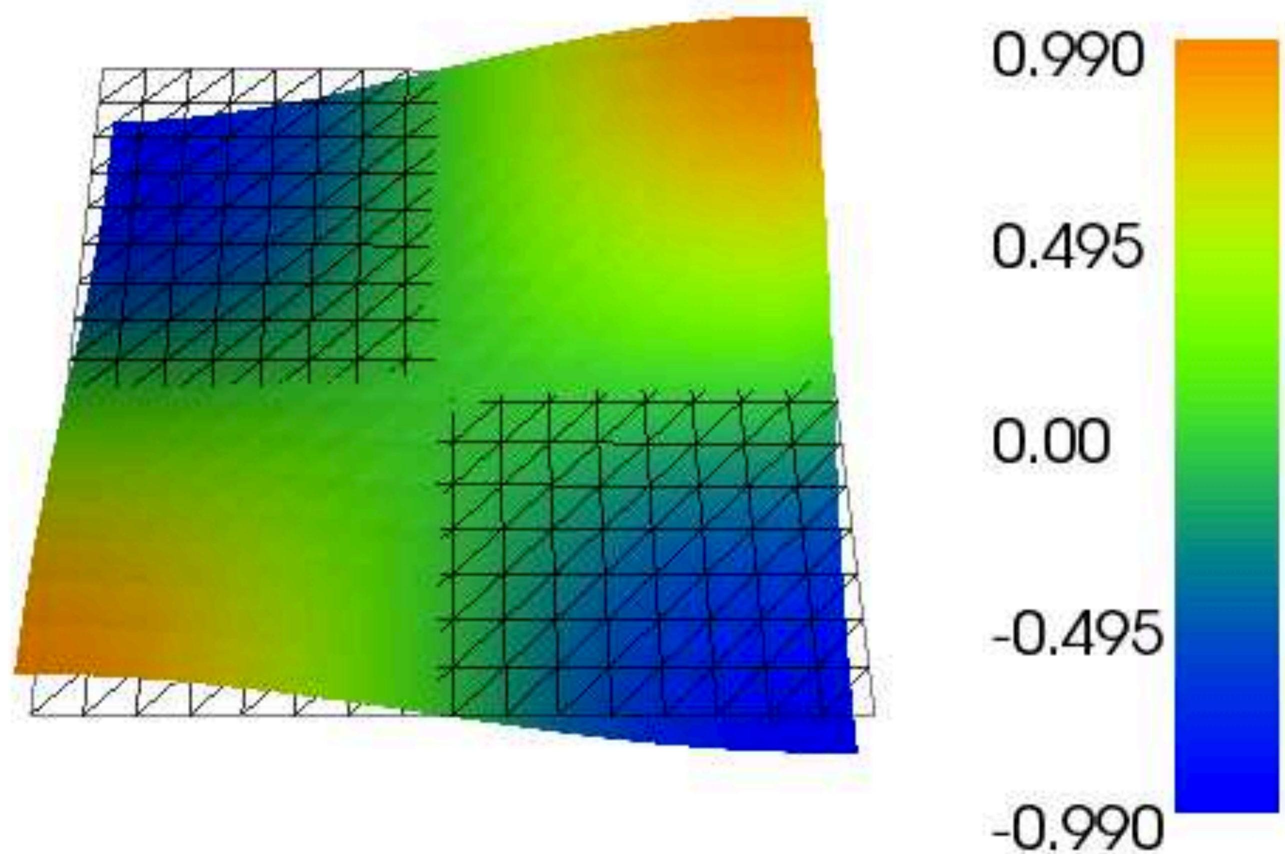


Figure 2: Mixed method using the spaces BDM(1) for $\underset{\sim}{V}_h$ and DG(0) for Π_h to approximate the boundary value problem (33).

Test problem

Let $\Omega = [0,1]^2$, and let

$$\Gamma = \{(x,y) \in \partial\Omega \ : \ x = 0 \text{ or } x = 1\}\,.$$

Consider the problem

$$\begin{aligned} -\Delta p &= 2\pi^2(\cos\pi x)(\cos\pi y) \\ \frac{\partial p}{\partial n} &= 0 \text{ on } \partial\Omega\backslash\Gamma \\ p &= (1-2x)(\cos\pi y) \text{ on } \Gamma \end{aligned} \tag{33}$$

whose exact solution is $p(x,y) = (\cos\pi x)(\cos\pi y)$.

We reformulate this using variational equations in (23).

Test problem implementation

Using the spaces

$$\text{BDM(1) for } \underset{\sim}{V}_h,$$

together with the

essential boundary condition $\mathbf{v} \cdot \mathbf{n} = 0$ on $\partial\Omega \backslash \Gamma$, and

$$\text{DG(0) for } \Pi_h,$$

we obtained the result depicted in Figure 2.

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