Simulations using equation (12.16) have been compared with laboratory experiments [44]. Solutions of (12.16) can be approximated numerically [46] by writing it in the form

$$\Big(I - \frac{\partial^2}{\partial x^2}\Big)u_t = -u_x - 2uu_x = -\big(u + u^2\big)_x, \tag{12.17}$$

where I denotes the identity operator. The left-hand side is a familiar elliptic operator, so we can write a variational equation

$$a(u_t, v) = b(u, v) \quad \text{for all } v \in V, \tag{12.18}$$

where

$$a(v, w) = \int_\Omega v(x)w(x) + v'(x)w'(x)\,dx \tag{12.19}$$

and

$$\begin{aligned} b(v, w) &= -\int_\Omega (v + v^2)'(x)w(x)\,dx = -\int_\Omega (v' + 2vv')(x)w(x)\,dx \\ &= -\int_\Omega (1 + 2v)(x)v'(x)w(x)\,dx. \end{aligned} \tag{12.20}$$

12.4.1 Time-stepping schemes

A variety of time-stepping schemes can be used to discretize (12.18). One family of schemes that work well [46] is the **Runge-Kutta** schemes. These schemes have the useful feature that they can be high order yet not require previous time values of the solution. They are often used as start-up schemes for other schemes that do utilize previous values, as do the θ schemes (12.11). The simplest of these schemes is often called the **modified Euler** scheme:

$$\begin{aligned} &a(\hat{u}^{n+1}, v) = a(u^n, v) - \Delta t\, b(u^n, v)\ \forall v \in V, \\ &a(u^{n+1}, v) = a(u^n, v) + \tfrac{1}{2}\Delta t\big(b(u^n, v) + b(\hat{u}^{n+1}, v)\big)\ \forall v \in V. \end{aligned} \tag{12.21}$$

The first step in (12.21) is called the **predictor step** and the second step in (12.21) is called the **corrector step**.

12.4.2 An example

In Figure 12.3, we present an example of the use of this scheme with piecewise linear approximation in space. This was computed using Program 12.2. The initial data $u(x, 0) = e^{-x^2}$ evolves into a complex wave form that spreads out as it travels to the right, and even has a part that moves slowly to the left. The wave speed is approximately $1 + 2u$, and in 25 time units the leading wave moves over 60 units to the right. A linear wave (that is, if u were small) would have moved only 25 units to the right.

A common measure of wave length for a complex wave shape is its width at half height. For the function $f(x) = e^{-x^2}$, this is the point x where $e^{-x^2} = 1/2$. Thus $-x^2 = \log(1/2)$, so $x = \sqrt{\log 2} = 0.83\ldots$. Thus the initial wave length is less than 2, whereas the leading wave at $T = 25$ has a width that is about 7 units, over 4 times larger. The dispersion causes the initial narrow wave to spread.