There is a subtle dependence of the regularity of the solution in the case of discontinuous coefficients [219]. It is not in general the case that the gradient of the solution is bounded. However, from the variational derivation, we see that the gradient of the solution is always square integrable. A bit more is true, that is, the p-th power of the solution is integrable for $2 \leq p \leq P_\varepsilon$ where P_ε is a number bigger than two depending only on the ellipticity constant ε in (17.7) (as ε tends to zero, P_ε tends to two).

17.1.1 Coercivity and continuity

The assumption (17.5) immediately implies coercivity of the bilinear form (17.3). For each $\mathbf{x} \in \Omega$, we take $\xi_i = v_{,i}(\mathbf{x})$ and apply (17.5):

$$\int_\Omega |\nabla \mathbf{v}(\mathbf{x})|^2 \, d\mathbf{x} = \int_\Omega \sum_{i=1}^d v_{,i}(\mathbf{x})^2 \, d\mathbf{x} \leq c_0 \int_\Omega \sum_{i,j=1}^d \alpha_{ij}(\mathbf{x}) v_{,i}(\mathbf{x}) v_{,j}(\mathbf{x}) \, d\mathbf{x} = c_0 a_\alpha(v,v). \quad (17.8)$$

Similarly, (17.6) implies that the bilinear form (17.3) is bounded:

$$\begin{aligned} a_\alpha(u,v) &= \int_\Omega \sum_{i,j=1}^d \alpha_{ij}(\mathbf{x}) u_{,i}(\mathbf{x}) v_{,j}(\mathbf{x}) \, d\mathbf{x} \leq c_1 \int_\Omega |\nabla u(\mathbf{x})| \, |\nabla v(\mathbf{x})| \, d\mathbf{x} \\ &\leq c_1 \|u\|_{H^1(\Omega)} \|v\|_{H^1(\Omega)}, \end{aligned} \quad (17.9)$$

using the Cauchy-Schwarz inequality (3.15).

17.1.2 Flux continuity

Using the variational form (17.3) of the equation (17.1), we will see that the *flux*

$$\sum_{i=1}^d \alpha_{ij}(\mathbf{x}) \frac{\partial u}{\partial x_i}(\mathbf{x}) n_j \quad (17.10)$$

is continuous across an interface normal to $\mathbf{n}$ even when the α_{ij}'s are discontinuous across the interface. This implies that the normal slope of the solution must have a jump (that is, the graph has a kink).

The derivation of (17.10) is just integration by parts. Suppose that $\Omega = \Omega_1 \cup \Omega_2$ and that the coefficients are smooth on the interiors of Ω_i, $i = 1, 2$, but have a jump across $\Gamma = \overline{\Omega_1} \cap \overline{\Omega_2}$.[1] Suppose that $v = 0$ on $\partial\Omega$. Define $\mathbf{w} = v\boldsymbol{\alpha}\nabla\mathbf{u}$ and apply the divergence theorem on each Ω_i separately to get

$$\oint_\Gamma v\mathbf{n}_i \cdot \boldsymbol{\alpha}\nabla\mathbf{u} \, ds = \int_{\Omega_i} \nabla\cdot \mathbf{w} \, dx = \int_{\Omega_i} \big(\boldsymbol{\alpha}\nabla\mathbf{u}\big) \cdot \nabla v \, dx + \int_{\Omega_i} v\nabla\cdot \big(\boldsymbol{\alpha}\nabla\mathbf{u}\big) \, dx$$

Summing this over i and using (17.1) we get

$$\oint_\Gamma v[\mathbf{n} \cdot \boldsymbol{\alpha}\nabla\mathbf{u}]_\Gamma \, ds = a(u,v) - \int_\Omega fv \, d\mathbf{x} = 0, \quad (17.11)$$

[1] We are being a bit picky here about whether the sets Ω_i include their boundaries (that is, are closed) or not. To write $\Omega = \Omega_1 \cup \Omega_2$, one of the Ω_i's has to include the overlap Γ.