Then for small C we expect that

$$u \approx Cv. \tag{9.4}$$

We see in Figure 11.2 that this is the case.

In this case, we can go one step further since it is possible to solve (9.3) exactly:

$$v(\xi) = \frac{1}{4} - \frac{e^{-2(\theta-\xi)} + e^{2(\theta-\xi)}}{4(e^{-2\theta} + e^{2\theta})}. \tag{9.5}$$

Therefore we can compute $\|u - Cv\|_{L^2(\Omega)}$ to see if it goes to zero as $C \to 0$.

Thus there are useful tests that can be done for nonlinear problems that can determine whether or not the code is producing reasonable answers.

9.2.2 Nonlinear variational problems

We will introduce subsequently a problem whose variational formulation takes the form

$$a(u,v) + n(u,v) = (f,v)_{L^2([0,1])} \quad \forall v \in V \tag{9.6}$$

where

$$a(u,v) = \int_0^\theta u'(x)v'(x) + 4u(x)v(x)\,dx$$

and the nonlinearity has been separated for convenience in the form

$$n(u,v) = 6\int_0^\theta u(x)^2 v(x)\,dx = 6\,(u^2,v)_{L^2([0,1])}. \tag{9.7}$$

Note that this is nonlinear in the trial function u and linear in the test function v.

A Galerkin method for a space with a basis $\{\phi_i : i = 1,\ldots,n\}$ can be written as a system of nonlinear equations

$$F_i(u) := a(u,\phi_i) + n(u,\phi_i) - (f,\phi_i)_{L^2([0,1])} = 0. \tag{9.8}$$

Writing $u = \sum_j U_j\phi_j$, Newton's method for this system of equations for (U_j) can be derived. However, it can also be cast in variational form as follows.

Instead of using a basis function, let us define a function F with coordinates parameterized by an arbitrary $v \in V$:

$$F_v(u) := a(u,v) + n(u,v) - (f,v)_{L^2([0,1])} \tag{9.9}$$

If $v = \phi_i$ then of course we have the previous function. Newton's method requires us to compute the derivative of F with respect to its "coordinates" which in this case correspond to elements of V. The derivative of F_v at u in the direction of $w \in V$ is, as always, a limit of a difference quotient,

$$\frac{F_v(u+\epsilon w) - F_v(u)}{\epsilon}, \tag{9.10}$$

as $\epsilon \to 0$. Expanding, we find that

$$\begin{aligned} F_v(u+\epsilon w) - F_v(u) &= \epsilon a(w,v) + 6\left((u+\epsilon w)^2 - u^2, v\right)_{L^2([0,1])} \\ &= \epsilon a(w,v) + 6\left(2\epsilon uw + \epsilon^2 w^2, v\right)_{L^2([0,1])}. \end{aligned} \tag{9.11}$$