n	Integral	Error
3	1.15**3**84615384615	8.5×10^{-4}
5	1.154**6**9613259669	4.4×10^{-6}
7	1.1547005**1**566839	2.3×10^{-8}
9	1.15470053**8**26218	1.2×10^{-10}
11	1.154700538378**6**5	6.0×10^{-13}

Table 13.1 Errors in computing the integral (13.28) via the trapezoidal rule with n points. The exact answer is 1.15470053837925, which is obtained with $n = 13$ and does not change for larger n. The bold face digits are the first incorrect digits for each n.

13.2 PEANO KERNEL THEOREM

There is a general abstract result due to Peano[4] that gives a representation of the error for a wide class of numerical approximations. The error in quadrature is a typical example. Consider the setup in theorem 13.2 and define

$$Ef = Qf - \int_a^b f(x)w(x)\,dx. \tag{13.29}$$

Note that $EP = 0$ for all polynomials of degree k, where k is the order of exactness of Q, and that E is linear,

$$E(f + cg) = Ef + cEg, \tag{13.30}$$

as long as the same is true of Q, since this holds for the integral. In particular, $Ef = E(f - P)$ for any polynomial P of degree k.

Recall Taylor's theorem with integral remainder (7.81):

$$f(x) - P_k(x) = \frac{1}{k!}\int_a^x (x-t)^k f^{(k+1)}(t)\,dt\,, \tag{13.31}$$

where P_k is the Taylor polynomial

$$P_k(x) = \sum_{j=0}^{k} \frac{f^{(j)}(a)}{j!}(x-a)^j. \tag{13.32}$$

Let us use the notation $(X)_+$ to mean X if $X \geq 0$ and 0 if $X \leq 0$. Then we can rewrite (13.31) as

$$f(x) - P_k(x) = \frac{1}{k!}\int_a^b (x-t)_+^k f^{(k+1)}(t)\,dt\,. \tag{13.33}$$

Since E is linear, we have

$$\begin{aligned} Ef = E(f-P) &= \frac{1}{k!}E\left[\int_a^b (x-t)_+^k f^{(k+1)}(t)\,dt\right] \\ &= \frac{1}{k!}\int_a^b E\left[(x-t)_+^k\right] f^{(k+1)}(t)\,dt. \end{aligned} \tag{13.34}$$

[4]Giuseppe Peano (1858–1932) is best known for his contributions to the foundations of mathematics. But he also did research on numerical analysis [130].

The last equality may seem like a leap of faith, and in any case the notation needs to be made more precise. Define

$$\phi(x) = \int_a^b (x-t)_+^k f^{(k+1)}(t)\,dt \tag{13.35}$$

for $x \in [a,b]$. Then (13.33) says that $f - P_k = (k!)^{-1}\phi$, so $Ef = (k!)^{-1}E\phi$. Similarly, define a one-parameter family of functions $\psi_t^k(x) = (x-t)_+^k$ for $x \in [a,b]$ and let

$$K(t) = E\psi_t^k. \tag{13.36}$$

Then we claim that

$$Ef = \int_a^b K(t) f^{(k+1)}(t)\,dt. \tag{13.37}$$

13.2.1 Continuity of Peano kernels

To make sense of the integral in (13.37), we need to know some regularity properties of K. Let us assume that Qf is defined for any $f \in C^m([a,b])$ for some $m \geq 0$. More precisely, we assume that there is a positive constant $C_Q < \infty$ such that

$$|Qf| \leq C_Q \|f\|_{C^m,[a,b]} \tag{13.38}$$

for all $f \in C^m([a,b])$, where

$$\|f\|_{C^m,[a,b]} = \max_{0\leq i\leq m} \|f^{(i)}\|_{\infty,[a,b]}. \tag{13.39}$$

In particular, we can take $m = 0$ for trapezoidal rule, $m = 1$ for the Hermite rule, and $m = 2k-1$ for the Euler-Maclaurin quadrature rule using k end corrections ($k = 1$ is the Hermite case). Note that (13.38) implies that

$$|Ef| \leq (C_Q + (b-a))\|f\|_{C^m,[a,b]}. \tag{13.40}$$

Then

$$\begin{aligned} |K(t+h) - K(t)| &= |E\psi_{t+h}^k - E\psi_t^k| = |E(\psi_{t+h}^k - \psi_t^k)| \\ &\leq (C_Q + (b-a))\|\psi_{t+h}^k - \psi_t^k\|_{C^m,[a,b]} \to 0 \end{aligned} \tag{13.41}$$

as $h \to 0$, provided $m < k$. In fact, it is sufficient to show that

$$\|\psi_{t+h}^k - \psi_t^k\|_{C^0,[a,b]} \to 0 \text{ as } h \to 0, \tag{13.42}$$

for $k > 0$, since $(\psi_t^k)' = k\psi_t^{k-1}$ for $k > 1$. We leave the proof of (13.42) as exercise 13.20. This shows that K is continuous.

The proof of (13.37) relies on the linearity of E and the linearity of the integration process. For example, this can be verified by approximating the integral by Riemann sums (exercise 13.6). Thus we have proved the following.

Theorem 13.5 *Suppose that the quadrature Q is linear, exact of order k, and satisfies the bound (13.38) for $m < k$. Then the error E defined by (13.29) satisfies*

$$Ef = \frac{1}{k!}\int_a^b K(t) f^{(k+1)}(t)\,dt, \tag{13.43}$$

where K is defined by (13.36).

The function K is called the *Peano kernel* for this error relation. We can provide an error estimate using the Peano kernel:

$$|Ef| \le \frac{1}{k!}\int_a^b |K(t)|\,dt\,\|f^{(k+1)}\|_{\infty,[a,b]}, \tag{13.44}$$

which can be compared with (13.5) (see exercise 13.7).

For $t \le x$, $\psi_t^k \equiv 0$, and so the kth derivative of ψ_t^k is discontinuous at $x = t$. However, it is easy to see that $\psi_t^k \in C^{k-1}(\mathbb{R})$ and

$$\begin{aligned} K'(t) &= \lim_{h\to 0} h^{-1}\left(K(t+h) - K(t)\right) = \lim_{h\to 0} h^{-1}\left(E\psi_{t+h}^k - E\psi_t^k\right) \\ &= \lim_{h\to 0} E\left(h^{-1}\left(\psi_{t+h}^k - \psi_t^k\right)\right). \end{aligned} \tag{13.45}$$

Similar to (13.42), we can show (exercise 13.21) that

$$\|h^{-1}(\psi_{t+h}^k - \psi_t^k) - k\psi_t^{k-1}\|_{C^m,[a,b]} \to 0 \text{ as } h \to 0, \tag{13.46}$$

for $k \ge m + 2$. Therefore by (13.40)

$$\begin{aligned} K'(t) &= \lim_{h\to 0} E\left(h^{-1}\left(\psi_{t+h}^k - \psi_t^k\right)\right) = E\Big(\lim_{h\to 0} h^{-1}\left(\psi_{t+h}^k - \psi_t^k\right)\Big) \\ &= kE\left(\psi_t^{k-1}\right), \end{aligned} \tag{13.47}$$

provided that Q satisfies (13.38). By definition, $\psi_t^0(x)$ is the Heaviside function that is 0 for $x < t$ and 1 for $x > t$.

When $t = a$, $\psi_a^k(x) = x^k$ on $[a, b]$, so we have $K(a) = 0$ because Q is exact of order k. Similarly, when $t = b$, $\psi_b^k \equiv 0$ on $[a, b]$, so again $K(b) = 0$. Therefore, (13.45) implies that

$$K^{(i)}(a) = K^{(i)}(b) = 0 \tag{13.48}$$

for $i = 0, 1, \ldots, k-1-m$, provided that Qf is well-defined for $f \in C^m([a, b])$. In the case of the Hermite quadrature rule (13.21), we have $m = 1$.

13.2.2 Examples of Peano kernels

Now let us see if we can figure out what K might look like in examples. Let us start with Q = midpoint rule on $[0, 1]$, which is exact for polynomials of degree $k = 1$. In this case, the statement is

$$Ef = f(\tfrac{1}{2}) - \int_0^1 f(t)\,dt = \int_0^1 K_{\mathrm{MR}}(t) f^{(2)}(t)\,dt. \tag{13.49}$$

The quadrature rule $Qf = f(\frac{1}{2})$ is well-defined for $f \in C^0$, so we conclude from (13.45) that $K_{\mathrm{MR}} \in C^0$ and that K'_{MR} is defined for $x \neq \frac{1}{2}$ and bounded. Thus we can integrate by parts to find

$$Ef = f(\tfrac{1}{2}) - \int_0^1 f(t)\,dt = -\int_0^1 K_{\mathrm{MR}}^{(1)}(t) f^{(1)}(t)\,dt. \tag{13.50}$$

We can integrate by parts again, but we have to be careful since K_{MR} is not C^1. However, the only point where K_{MR} fails to be smooth is $x = \frac{1}{2}$, and so we can break the integral into two parts and integrate by parts again. To make a long story short, we find that

$$K_{\mathrm{MR}}(t) = -\begin{cases} \frac{1}{2}t^2 & t \leq \frac{1}{2} \\ \frac{1}{2}(t-1)^2 & t \geq \frac{1}{2}. \end{cases} \tag{13.51}$$

We leave as exercise 13.8 verification that this K_{MR} satisfies (13.49) for all $f \in C^2$. Similarly, it is not hard to see (exercise 13.7) that the kernel for the trapezoidal rule is

$$K_{\mathrm{TR}}(t) = \tfrac{1}{2}t(1-t) \tag{13.52}$$

and the kernel for Hermite quadrature (13.21) is

$$K_{\mathrm{H}}(x) = -\tfrac{1}{24}x^2(1-x)^2. \tag{13.53}$$

We will consider the form of the general kernels K_k^{EM} for the Euler-Maclaurin quadrature subsequently.

13.2.3 Uniqueness of Peano kernels

Suppose that there were two kernels K and $\widetilde{K}$ in $C^0[a,b]$ such that (13.43) holds. Then we claim that we must have $K = \widetilde{K}$. To prove this, we use (13.43) twice to see that

$$\int_a^b (K(t) - \widetilde{K}(t)) f^{(k+1)}(t)\,dt = 0 \tag{13.54}$$

for all $f \in C^{k+1}([a,b])$. For any $g \in C^0[a,b]$, we can write

$$f(x) = \int_a^x \int_a^t \cdots \int_a^s g(s)\,ds, \tag{13.55}$$

where there are $k+1$ integrals. Then we conclude that $g(x) = f^{(k+1)}(x)$ for all $x \in [a,b]$. Thus (13.54) implies

$$\int_a^b (K(t) - \widetilde{K}(t)) g(t)\,dt = 0 \tag{13.56}$$

for any $g \in C^0[a,b]$. Define $e(t) = K(t) - \widetilde{K}(t)$ for $t \in [a,b]$. Suppose that there is some $t_0 \in [a,b]$ such that $e(t_0) \neq 0$. Without loss of generality, we can assume that $a < t_0 < b$, because if $e(a) \neq 0$ then by continuity of e we must have $e(t) \neq 0$ for some $t > a$, and the analog would hold if

$e(b) \neq 0$. Then there are some $\epsilon > 0$ and $\delta > 0$ such that $e(t_0)e(t) \geq \delta$ for all $t \in [t_0 - \epsilon, t_0 + \epsilon] \subset [a, b]$. Define $g \in C^0[a, b]$ by

$$g(t) = \begin{cases} e(t_0)(\epsilon^2 - (t - t_0)^2) & |t - t_0| \leq \epsilon \\ 0 & |t - t_0| \geq \epsilon \end{cases}. \tag{13.57}$$

Then

$$\begin{aligned} \int_a^b (K(t) - \widetilde{K}(t))g(t)\,dt &= \int_{t_0-\epsilon}^{t_0+\epsilon} e(t)\,g(t)\,dt \\ &\geq \delta \int_{t_0-\epsilon}^{t_0+\epsilon} (\epsilon^2 - (t - t_0)^2)\,dt > 0, \end{aligned} \tag{13.58}$$

contradicting (13.56). Thus we must have $K(t) = \widetilde{K}(t)$ for all $t \in [a, b]$.

13.2.4 Composite Peano kernels

If we make a simple change of variables in the integration, the Peano kernel changes in a predictable way. Suppose that $\widehat{K}$ denotes the Peano kernel for the interval $[0, 1]$. Then the kernel for the interval $[a, a + h]$ is

$$K(a + ht) = h^k \widehat{K}(t), \tag{13.59}$$

where k is the order of exactness.

To see why this is so, we need to perform the corresponding transformations for both the integral and the quadrature rule. Define $g(x) = a + hx$. Then for $f : [a, a + h] \to \mathbb{R}$

$$\int_0^1 f \circ g(x)\,dx = h \int_a^{a+h} f(t)\,dt \tag{13.60}$$

Suppose that

$$Q_{[0,1]}(f \circ g(x)) = hQ_{[a,a+h]}(f). \tag{13.61}$$

Then

$$\begin{aligned} \frac{h^{k+1}}{k!} \int_0^1 \widehat{K}(t)(f^{(k+1)} \circ g)(t)\,dt &= \frac{1}{k!} \int_0^1 \widehat{K}(t)(f \circ g)^{(k+1)}(t)\,dt \\ &= E_{[0,1]}(f \circ g(x)) = hE_{[a,a+h]}(f) \\ &= \frac{h}{k!} \int_a^{a+h} K(t) f^{(k+1)}(t)\,dt, \end{aligned} \tag{13.62}$$

for any $f \in C^{k+1}([a, a + h]$, proving (13.59).

For the Euler-Maclaurin formula (13.25), we have

$$\begin{aligned} & h\left(\tfrac{1}{2}f(a) + \sum_{i=1}^{n-1} f(\xi_i) + \tfrac{1}{2}f(b)\right) + \sum_{i=1}^{k} c_i h^{2i}(f^{(2i-1)}(a) - f^{(2i-1)}(b)) \\ &= \int_a^b f(x)\,dx + h^{2k+3} \sum_{i=0}^{n-1} \int_0^1 K_k^{\text{EM}}(x) f^{(2k+2)}(a + h(i + x))\,dx. \end{aligned} \tag{13.63}$$

This completes the proof of theorem 13.4. The kernels K_k^{EM} are related to the Bernoulli polynomials [43, 102].