Basics of surface wave simulation

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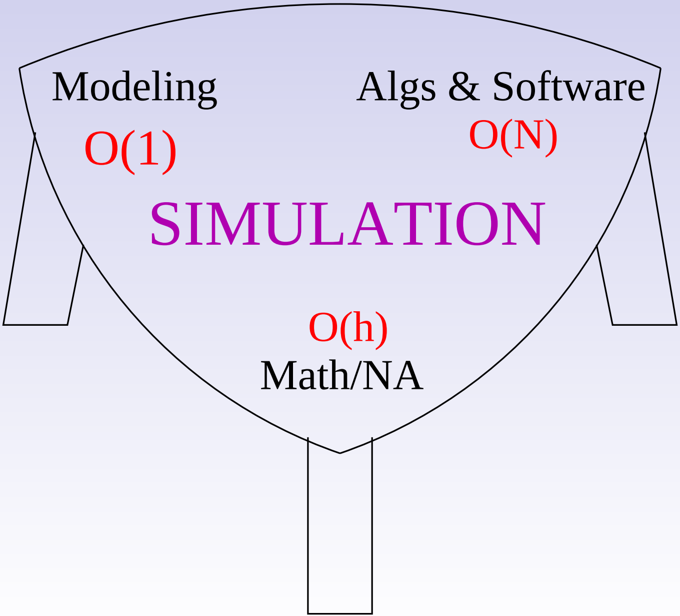
Components of simulation technology



Queen Charlotte quake

AP: Recent earthquake in Queen Charlotte Islands

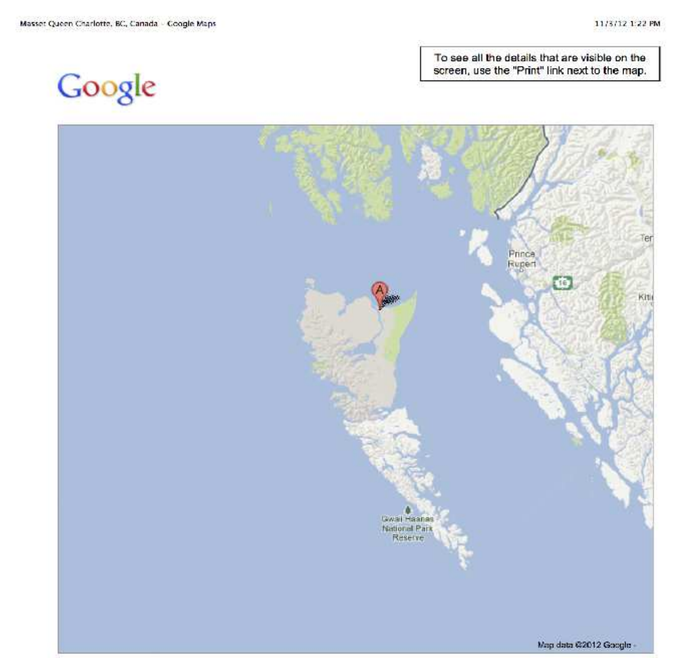
A 7.7 magnitude earthquake occurred at 8:04 pm (PST) October 27, 2012 near the Queen Charlotte Islands off the west coast of Canada, epicenter 155 kilometers (96 miles) south of Masset.

The Pacific Tsunami Warning Center announced that a tsunami wave was headed toward Hawaii and that the first tsunami wave could hit the islands by about 10:30 p.m. local time (1:30 am PST, 5.5 hours later).

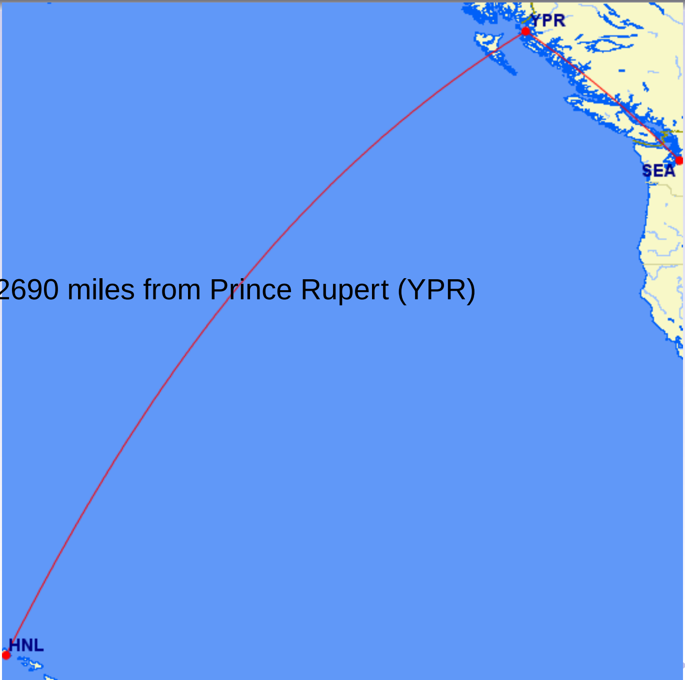
A 69-centimeter (27”) wave was recorded off Langara Island on the northeast tip of Haida Gwaii. Another 55 centimeter (21”) wave hit Winter Harbour on the northeast coast of Vancouver Island.

The Queen Charlotte Islands are also known by their official indigenous name of Haida Gwaii. Comprising about 150 islands located north of Canada’s Vancouver Island, their total population is about 5,000 of which the Haida people make up about 45%.

Where is Masset

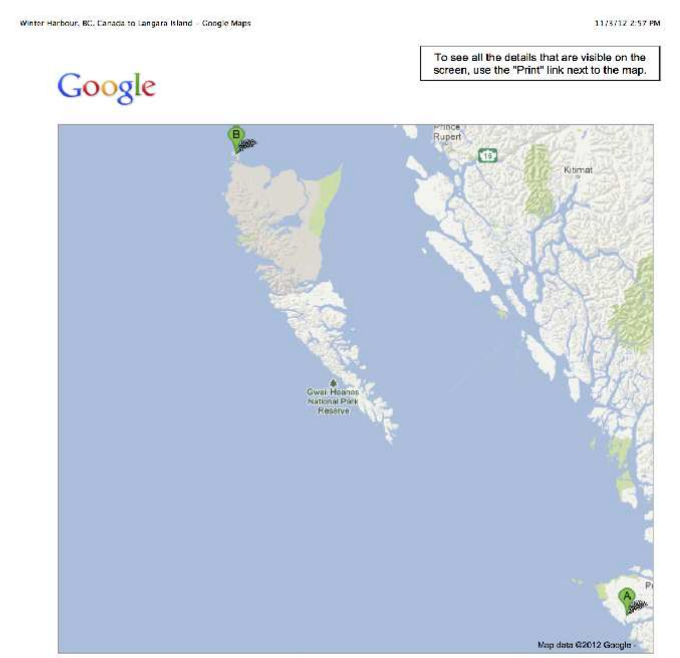


How far is Hawaii?



2690 miles from Prince Rupert (YPR)

Where the waves were measured



Conclusions drawn from the news:

- Tsunamis are not very big (less than a meter)
- But they move very fast (close to the speed of sound)
- They can travel far (around the world) and still be a threat

But how long are the waves?

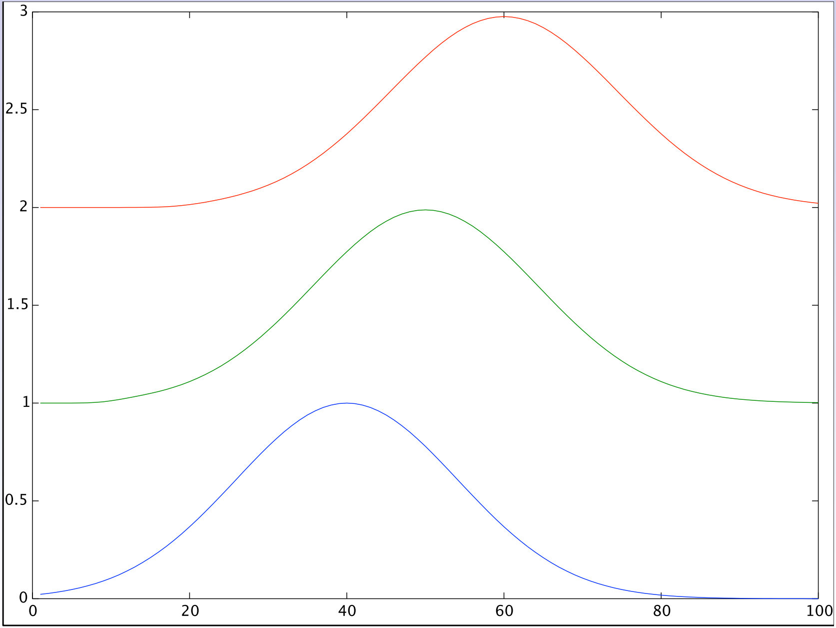
A very long wave with small amplitude can carry a great deal of energy!

Water motion is multifactorial

- advection
- nonlinearity
- shocks
- dissipation
- dispersion

We will study how each of these relates to numerical methods

Advection: things that move



Advection model equation

Simple advection relates changes in time with changes in space:

$$u_t + cu_x = 0$$

Solutions to this equation satisfy

$$u(t, x) = v(x - ct)$$

The proof is simple:

$$u_x = v' \qquad u_t = -cv'$$

Things just move to the right at speed c.

Nonlinear advection model equation

Some physical quantities satisfy a nonlinear advection equation:

$$0 = u_t + f(u)_x = u_t + f'(u)u_x$$

Solutions no longer just translate to the right:

$$u(t,x) \neq v(x - ct)$$

Things move to the right at speeds ($c(t,x) = f'(u)$) that depend on the size of u and they can change shape.

We can see what happens computationally in the case is $f(u) = u^2$.

Finite difference approximation

We can approximate u on a grid in space and time:

$$u(i\Delta t, j\Delta x) \approx u_{i,j}$$

We write

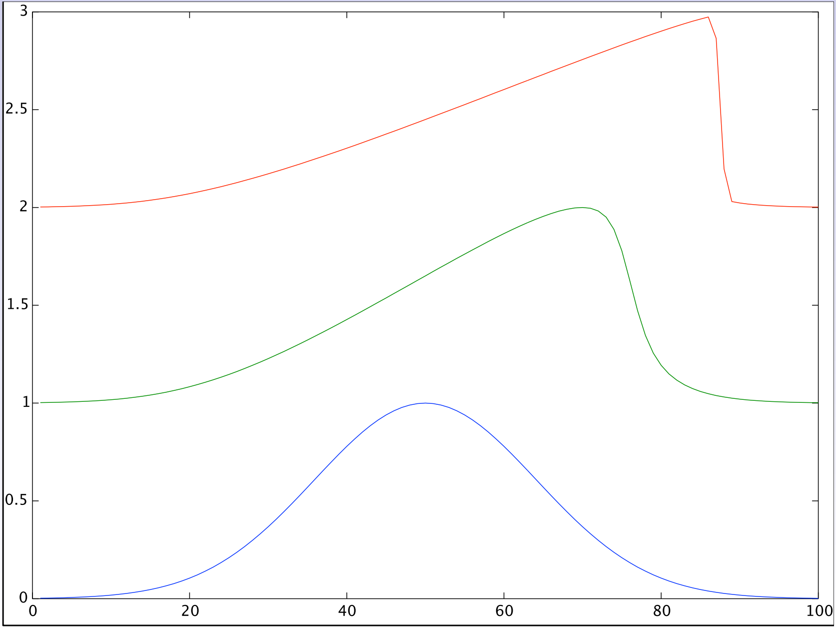
$$u_t(i\Delta t, j\Delta x) \approx \frac{u_{i,j} - u_{i-1,j}}{\Delta t}$$

$$f(u)_x(i\Delta t, j\Delta x) \approx \frac{f(u)_{i,j} - f(u)_{i,j-1}}{\Delta x}$$

Thus we obtain an algorithm

$$u_{i+1,j} = u_{i,j} - \frac{\Delta t}{\Delta x}\left(f(u)_{i,j} - f(u)_{i,j-1}\right)$$

Nonlinearity: things change shape ($f(u) = u^2$)



Shock formation

In the nonlinear advection case, we see that a discontinuity (shock) can form.
But the integral of u is preserved: integrating the advection equation in space (and integrating by parts) gives

$$\left(\int u\,dx\right)_t = \int u_t\,dx = -\int f(u)_x\,dx = 0. \qquad (1)$$

Thus the area under the graph of u is constant, and so its amplitude must decrease.

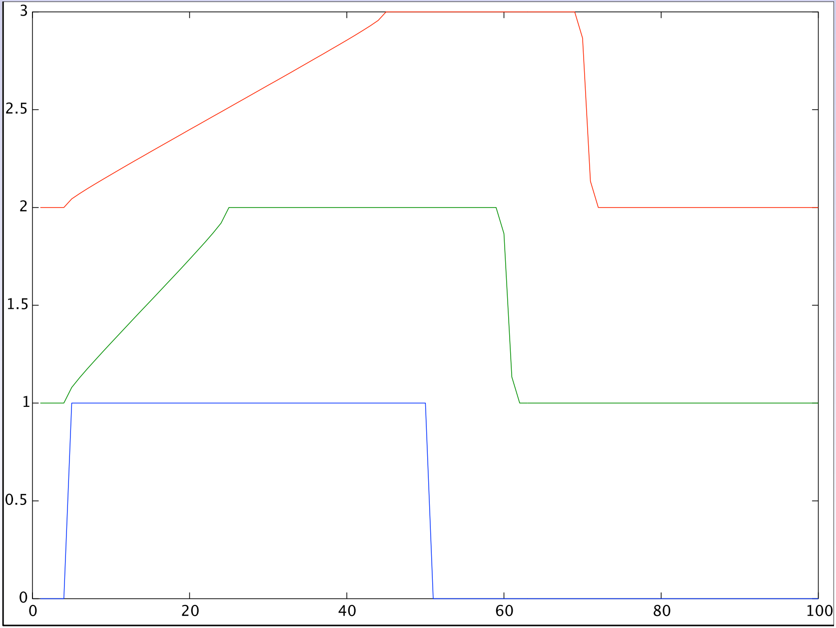
Shock properties

The integral of u^2 is also preserved: multiplying the advection equation by u and integrating in space (and integrating by parts) gives

$$\begin{aligned}\frac{1}{2}\left(\int u^2\,dx\right)_t &= \int uu_t\,dx = -\int f(u)_x u\,dx \\ &= \int f(u)u_x\,dx = \int g(u)_x\,dx = 0\end{aligned} \tag{2}$$

where $g' = f$ and g is an antiderivative of f with $g(0) = 0$.

Shocks: discontinuities that move

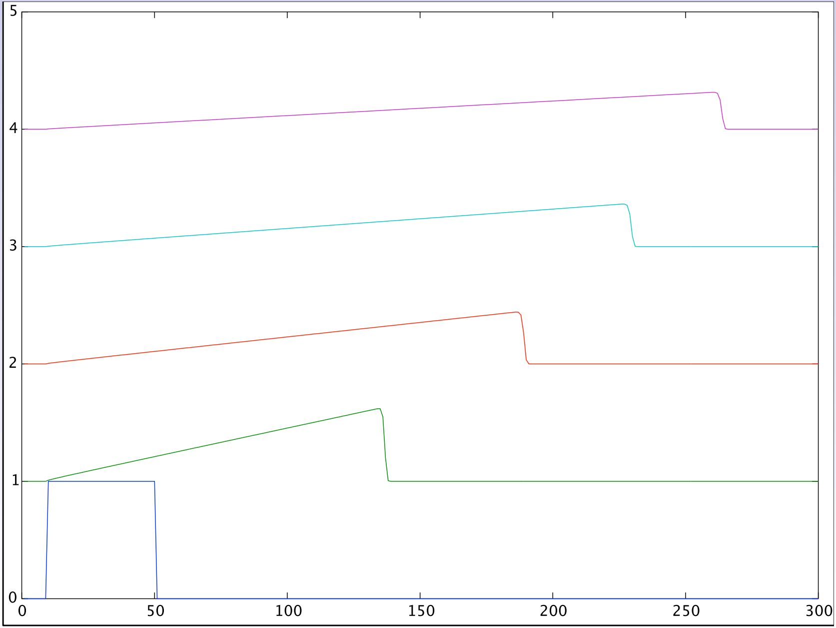


Shock fronts stay sharp, but back remains continuous.

The amplitude has to decrease since the integrals of u and u^2 remain constant.

Over time, the wave amplitude goes to zero.

Long-time development of shocks



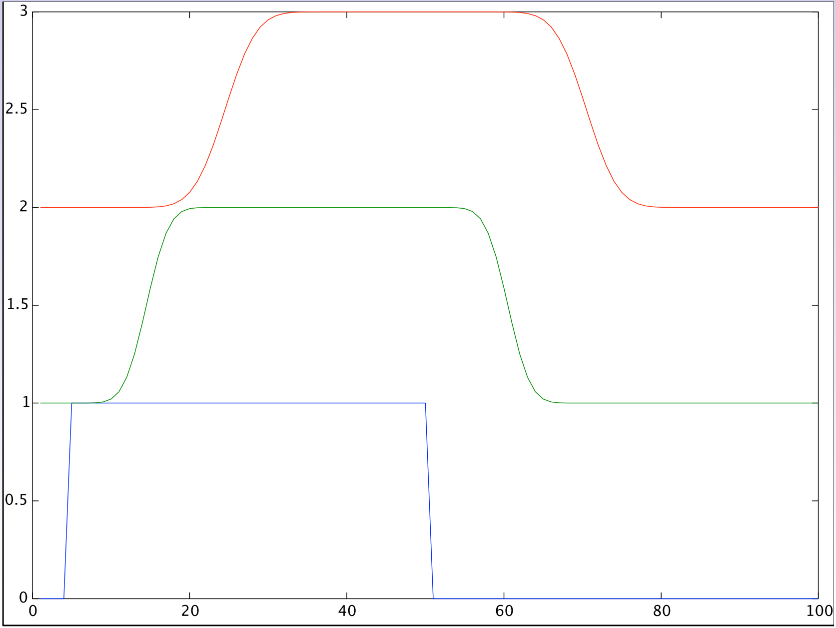
In the linear case, even discontinuous solutions are propagated by translation:

$$u(t,x) = v(x - ct)$$

Thus the linear case is quite different from the nonlinear case.

Even though the exact solution is trivial, let's see what our difference method produces.

Linear shocks: discontinuities that mush



Linear versus nonlinear shocks

Discontinuous solutions do propagate by translation:

$$u(t,x) \approx v(x-ct)$$

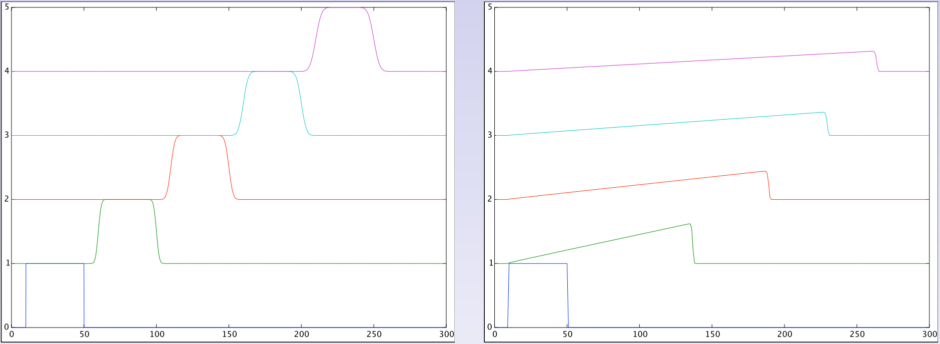
but the sharp edges are smoothed off.

We see an artifact of the numerical approximation.

We did not see this with smooth solutions or even with discontinuous solutions for nonlinear advection.

We need to understand what is going wrong.

Linear versus nonlinear shocks



linear advection: left .... nonlinear advection: right.

Suggests nonlinearity controls diffusion artifacts.

Harten advocated artificial compression [Sod78].

Finite difference approximation reviewed

Taylor's approximation says

$$\frac{u_{i,j} - u_{i,j-1}}{\Delta x} \approx u_x(i\Delta t, j\Delta x) + \frac{\Delta x}{2} u_{xx}(i\Delta t, j\Delta x) \quad (3)$$

Thus the difference scheme is actually a better approximation to

$$u_t + u_x - \frac{\Delta x}{2} u_{xx} = 0$$

than it is to the advection equation

$$u_t + u_x = 0$$

Numerical dissipation

The second-order derivative term in (Burger's equation)

$$u_t + f(u)_x - \epsilon u_{xx} = 0$$

is called a dissipation term due to the following. Multiply the equation by u, integrate in space and integrate by parts to get

$$\frac{1}{2}\left(\int u^2\,dx\right)_t + \epsilon \int u_x^2\,dx = 0 \qquad (4)$$

in view of (2).
Now we see that the integral of u^2 must dissipate to zero.

Numerical errors

It is possible to reduce numerical dissipation, but not eliminate it [CH78].
For example, the Lax-Wendroff scheme is

$$u_{i+1,j} = \sum_{k=-1}^{1} b_k u_{i,j}$$

where $b_{\pm 1} = \frac{1}{2}\alpha(\alpha \pm 1)$ and $b_0 = 1 - \alpha^2$, where $\alpha = \Delta t / \Delta x$ is the CFL number, is a better approximation to

$$u_t + u_x - \gamma \Delta x^2 u_{xxx} = 0$$

Exercise: compute γ.

Numerical dispersion

The third-order derivative term in

$$u_t + f(u)_x - \epsilon u_{xxx} = 0$$

is called a dispersion term.
Multiply the dispersion term by u, integrate in space and integrate by parts to get

$$\begin{aligned} \int u u_{xxx}\, dx &= -\int u_x u_{xx}\, dx \\ &= -\int \tfrac{1}{2}((u_x)^2)_x\, dx = 0 \end{aligned} \tag{5}$$

In view of (2), we conclude that the integral of u^2 is conserved.

KdV equation

The equation balances nonlinear advection with dispersion:

$$u_t + 6uu_x + u_{xxx} = 0 \qquad (6)$$

(Korteweg & de Vries 1895, Boussinesq [Bou77, p. 360]); has a family of solutions

$$u(t,x) = \frac{c}{2}\text{sech}^2\left(\tfrac{1}{2}\sqrt{c}(x - ct)\right)$$

which move at constant speed c without change of shape.

Matches observations of J. Scott Russell (1845).

BBM equation

An equivalent equation that balances nonlinear advection with dispersion is

$$u_t + u_x + 2uu_x - u_{xxt} = 0 \tag{7}$$

(Peregrine 1964, Benjamin, Bona and Mahoney 1972) which has similar solutions [ZWG02]

$$u(t,x) = \frac{3}{2} a \,\mathrm{sech}^2 \left(\frac{1}{2} \sqrt{\frac{a}{a+1}} \left(x - (1+a)t \right) \right)$$

The BBM equation is better behaved numerically.

$$u_t = -\left(1 - \frac{d}{dx^2}\right)^{-1} \frac{d}{dx} \left(u + u^2\right) = B\left(u + u^2\right) \tag{8}$$

Solitary wave exercises

The KdV and BBM equations can be compared by using the underlying advection model $u_t + u_x = 0$.

Thus we can swap time derivatives for (minus) space derivatives: $u_t \approx -u_x$. This suggests the near equivalence of the terms $u_{xxx} \approx -u_{xxt}$.

Derive the solitary wave solution for

$$u_t + u_x + 2uu_x + u_{xxx} = 0 \tag{9}$$

and compare this with the solitary wave for BBM

Show that the two forms converge as the wave amplitude goes to zero.

Tsunami controversy

Terry Tao says ”solitons are large-amplitude (and thus nonlinear) phenomena, whereas tsunami propagation (in deep water, at least) is governed by low-amplitude (and thus essentially linear) equations. Typically, linear waves disperse due to the fact that the group velocity is usually sensitive to the wavelength; but in the tsunami regime, the group velocity is driven by pressure effects that relate to the depth of the ocean rather than the wavelength of the wave, and as such there is essentially no dispersion, thus creating traveling waves that have some superficial resemblance to solitons, but arise through a different mechanism.
It is true, though, that KdV also arises from a shallow water wave approximation. The main distinction seems to be that the shallow water equation comes from assuming that the pressure behaves like the hydrostatic pressure, whereas KdV arises if one assumes instead that the velocity is irrotational (which is definitely not the case for tsunami waves).”

Tsunami analysis

We know that tsunami propagation requires long wave lengths since the amplitude is small.

Otherwise, no devastating amount of energy can be transmitted.

The time scale of tsunami impact is minutes, not hours as occurs in hurricane storm surge.

So the wave needs to be long and fast.

KDV/BBM provide such a mechanism.

Key question: what causes such a long wave to form?

Modeling question: does KdV require flow to be irrotational?

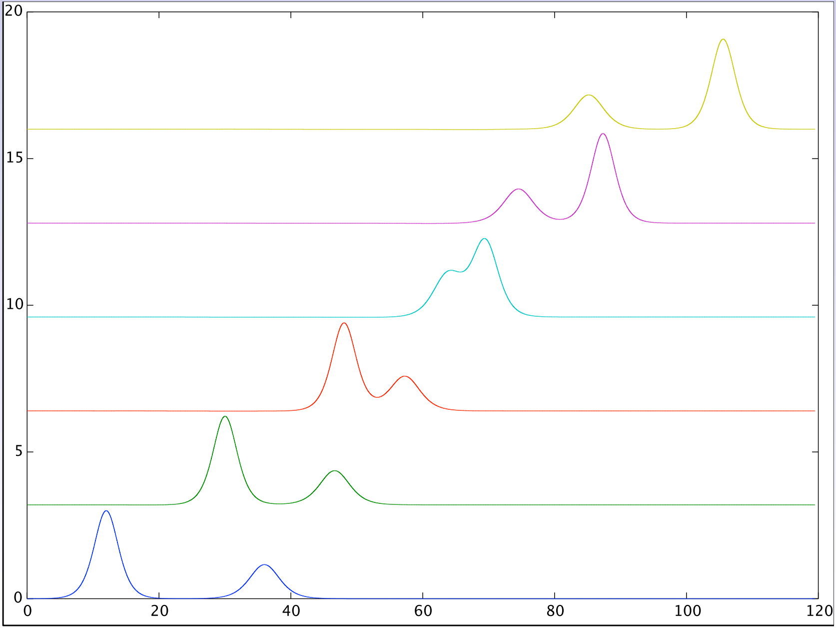
There are many other types of solutions to KdV/BBM.

- soliton interactions
- dispersion
- compare: no dispersion
- dispersive shock waves [EKL12]

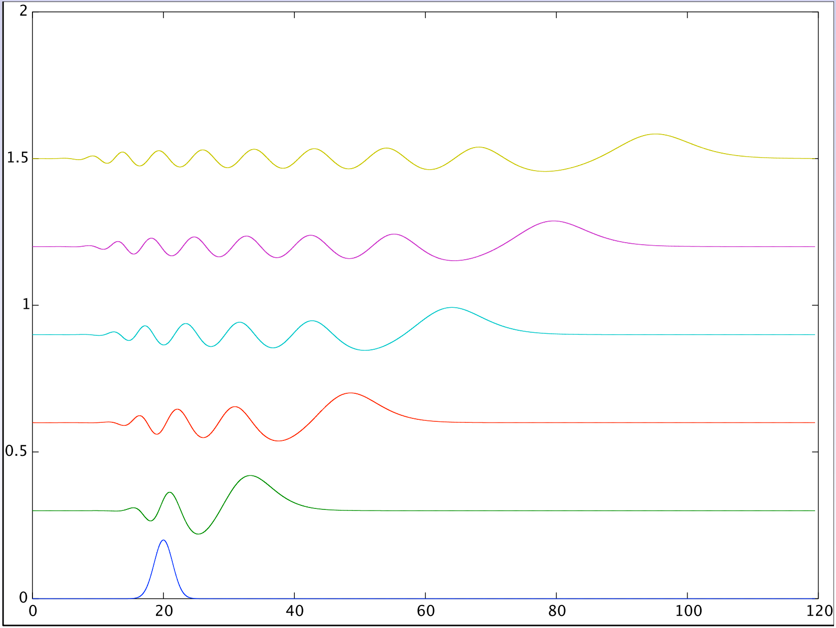
Exercise: explore different initial states

Compare with data [Gre61].

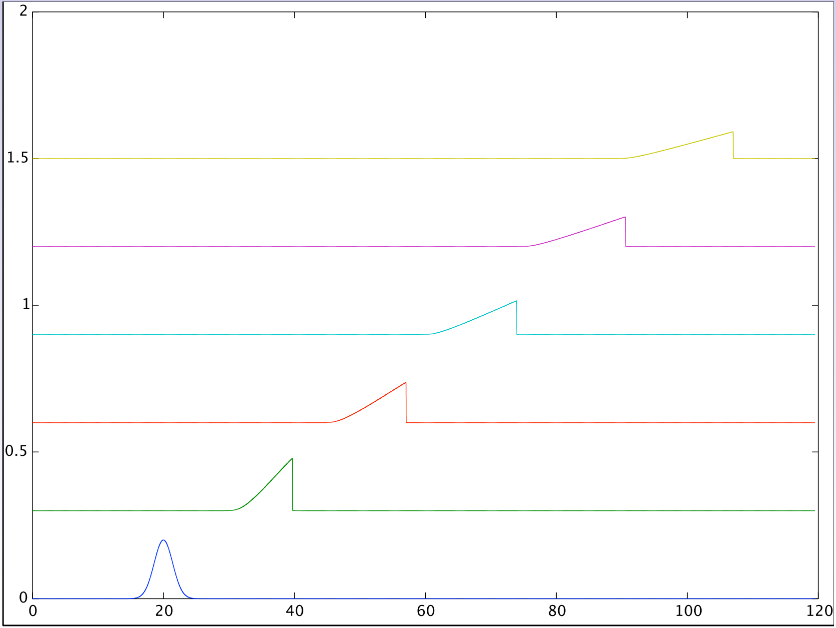
Soliton interaction (BBM)



Gaussian dispersion (BBM)



Compare Gaussian with no dispersion



Software issues

One-D problems are simple,

so you can use simple software systems, e.g., Matlab/octave.

Consider the time-stepping scheme for the advection problem

$$0 = u_t + f(u)_x$$

given by

$$u_{i+1,j} = u_{i,j} - \frac{\Delta t}{\Delta x}\left(f(u)_{i,j} - f(u)_{i,j-1}\right)$$

Tricks with octave: filter

The “filter” command performs finite difference specified by vectors “b” and “a”:

```
b=[ +1 -1 ];
a=[ 1 ];
xr=dx*[1:1000000];
yu=exp(-(.05*(xr-50)).^ 2);
...
cfl=dt/dx
for k=1:nts
yu=yu-cfl*filter(b,a,yu + yu .* yu);
end
```

Details about filter

Typing “help filter” in octave produces

- - Loadable Function: y = filter (B, A, X)

Return the solution to the following linear, time-invariant difference equation:

$$\sum_{k=0}^{N} a(k+1)y(n-k) = \sum_{k=0}^{M} b(k+1)x(n-k)$$

where N=length(a)-1 and M=length(b)-1.

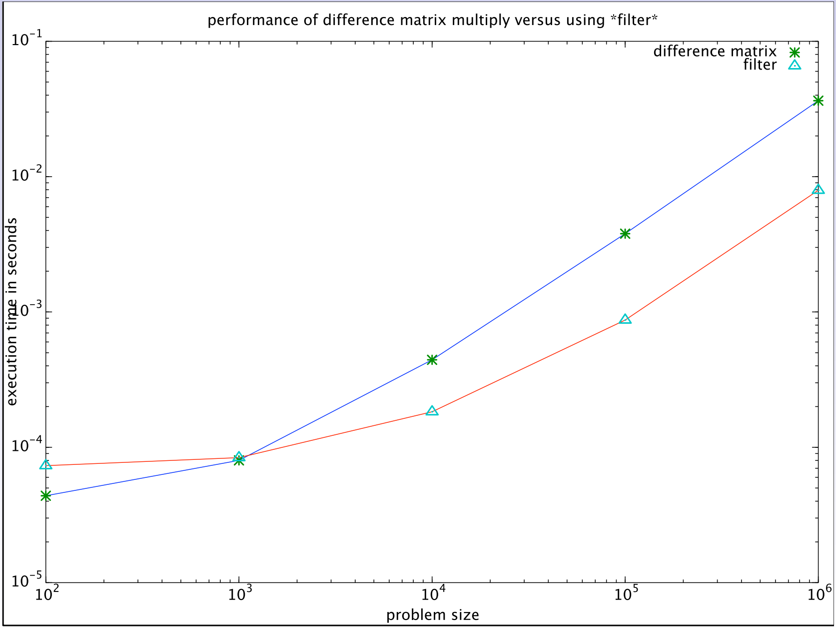
Equivalent difference matrix

Using “filter” is equivalent to multiplying by the sparse matrix “fod” defined as follows:

```
tdx=2*dx;
kc=0;
for k=2:nr;
kc=kc+1; hiv(kc)=k; hjv(kc)=k-1; hsv(kc)=-(1/tdx);
end
for k=1:nr;
kc=kc+1; hiv(kc)=k; hjv(kc)=k; hsv(kc)=+(1/tdx);
end
fod=sparse(hiv,hjv,hsv);
```

Key is to create a sparse matrix.

Performance of difference matrix vs. filter



Using filter for boundary value problems

There are some challenges is using “filter” to solve two-point boundary value problems.
Suppose we want to solve

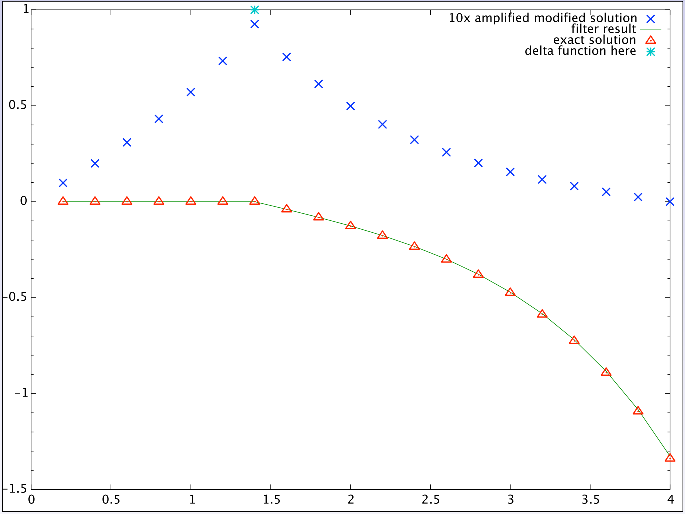
$$\alpha u - u_{xx} = f \text{ on } x_0 < x < x_1, \quad u(x_i) = 0 \text{ for } i = 0, 1.$$

We can do this via

```
b=[ 0 1 ];
a=[0 alfa 0]+(1/(dx*dx))*[-1 2 -1 ];
u=filter(b,a,f);
```

However, filter assumes a boundary condition $u(x_0) = u'(x_0) = 0$.

Behavior of filter



$u(x_0) = u'(x_0) = 0$ gives different solution
need to modify it by a homogeneous solution to get the correct boundary conditions.

Experimental comparisons

BBM model has been tested against laboratory experiments [BPS81]

Key parameter for model is the Stokes number

$$S = \frac{a\lambda^2}{d^3},$$

where

- a is the wave amplitude,
- λ is the wave length, and
- d is the water depth.

Example: $a = 1$, $\lambda = 10^6$, $d = 10^4$ (meters)
$\implies S = 1$.

Modeling issues

For $S < 1$, the data in [BPS81] suggest that the linear dispersive model is as accurate as nonlinear dispersive

For larger $S > 10$ the model experiences greater that 10% errors.

Question: how important is dispersion in such simulations?

The results in [BPS81] also suggest the importance of dissipation due to bottom friction for small values of depth d.

Comparing two models

What about the different nonlinear, dispersive models: KdV versus BBM?

Possible to give analytic comparisons [BPS83].

Compare the model [BC99]

$$u_t + u_x + 2uu_x + u_{xtt} = 0$$

Time scales

The time scale for these models is

$$t = \sqrt{\frac{d}{g}}$$

where d is the water depth and g is the acceleration due to gravity:

$$g = 9.81 \text{ meters/second}^2 \approx 32.2 \text{ feet/second}^2$$

For $d \approx 10^4$ meters, this means $t \approx \frac{1}{2}$ minute.
For $d \approx 10$ meters, this means $t \approx$ one second.
Thus we can think that the time scale of interest is a small number of seconds, less than a minute.

Wave speeds

For small amplitude waves, the wave speed in nondimensional coordinates is essentially 1.
That means the wave speed is the length scale divided by the time scale.
Therefore the speed c is given by

$$c \approx \frac{d}{t} = \frac{d}{\sqrt{d/g}} = \sqrt{dg}$$

For $d \approx 10^4$ meters, this means
$c \approx 313$ meters/second ≈ 700 miles/hour.
(speed of sound at sea level is 343.2 m/s)
For $d \approx 10$ m, $c \approx 9.9$ m/s ≈ 22 miles/hour.
For reference, Usain Bolt has run 100 meters at an average speed of 10.44 meters per second

Comparing nonlinearity and dispersion

Suppose we have a wave of amplitude $\alpha = a/d$ and wave length $\lambda = L/d$.

That is, $u(x) \approx \alpha\phi(x/\lambda)$. Then KdV looks like

$$u_t + u_x(1 + \alpha + \lambda^{-2}) = 0 \qquad (10)$$

We have seen that tsunamis have small amplitude: $\alpha \approx 10^{-4}$.

This suggests that nonlinearity has little effect.

But how big can the wave length be?

Known inundation by tsunamis places a limit on L.

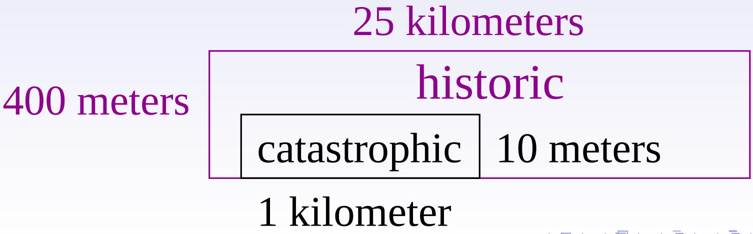
Wave length

The character of the wave propagation depends on wave length.

For a fixed mass of water, a smaller amplitude requires a longer wave length.

For a 1 meter wave, a length $L = 10$ kilometers $(\lambda = L/d = 1)$ yields a catastrophic wave.

Historic 10 meter tsunamis might have $\lambda/d = 100$.



Compare the 1960 Chilean-generated tsunami effect on Hilo

Hilo, HI – Google Maps 11/14/12 6:58 AM

To see all the details that are visible on the screen, use the "Print" link next to the map.

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