Classification: Mathematics

Post's Program and incomplete recursively enumerable sets

(recursive function theory/ Turing computability/ degrees of unsolvability)

by

Leo Harrington and Robert I. Soare

Department of Mathematics, University of California, Berkeley, CA, 94720
Department of Mathematics, University of Chicago, Chicago, IL, 60637

Abbreviations: r.e., recursively enumerable.

ABSTRACT:

A set A of nonnegative integers is *recursively enumerable (r.e.)* if A can be computably listed. It is shown that there is a first order property, $Q(X)$, definable in $\mathcal{E}$, the lattice of r.e. sets under inclusion, such that: (1) if A is any r.e. set satisfying $Q(A)$ then A is nonrecursive and Turing incomplete; and (2) there exists an r.e. set A satisfying $Q(A)$. This resolves a long open question stemming from Post's Program of 1944, and it sheds new light on the fundamental problem of the relationship between the algebraic structure of an r.e. set A and the (Turing) degree of information which A encodes.

Recursively enumerable (r.e.) sets have been a central topic in mathematical logic, in recursion theory (i.e. computability theory), and in undecidable problems. They are the next most effective type of set beyond recursive (i.e. computable) sets, and they occur naturally in many branches of mathematics. This together with the existence of nonrecursive r.e. sets has enabled them to play a key role in famous results such as Gödel's incompleteness theorem, the unsolvability of Hilbert's tenth problem on Diophantine equations, and the unsolvability of the word problem for finitely presented groups.

For sets $A, B \subseteq \omega$, A is *Turing reducible to* (also called *recursive in*) B, written $A \leq_T B$, if there is an algorithm for deciding whether $x \in A$ provided we are given answers to any questions of the form "Is $y \in B$?". We write $A \equiv_T B$ if $A \leq_T B$ and $B \leq_T A$. The equivalence class of A under $\equiv_T$ is the *(Turing) degree* (*degree of unsolvability*) of A, written $\deg(A) = \mathbf{a}$.

In his famous incompleteness theorem paper (1) Gödel defined a set of natural numbers which (in modern terminology) is the complement of the canonical nonrecursive r.e. set $K = \{e : e \in W_e\}$, where $\{W_e\}_{e\in\omega}$ is an acceptable numbering of all r.e. sets. Post (2) noted that K is (Turing) *complete* in the sense that $W_e \leq_T K$ for every W_e. In an attempt to classify the degree of unsolvability of problems in mathematics, Post (2) posed his famous question (*Post's Problem*) of whether there exist nonrecursive incomplete r.e. sets. (If not then there would be only one unsolvable problem for Diophantine equations, for finitely presented groups, and for many other problems in mathematics and computer science.)

Post's Program for resolving his problem was to find some property of an r.e. set A (not involving relative computability) which guarantees A nonrecursive and incomplete, i.e. $\emptyset <_T A <_T K$. In 1956 Myhill noted that the r.e. sets form a lattice $\mathcal{E}$ under inclusion, and $\mathcal{E}$ quickly became the standard setting in which to study the algebraic properties of r.e. sets. It became widely regarded that the desirable solution to Post's Program was to find an answer to the following.

Question 1 *Does there exist an $\mathcal{E}$-definable property, $P(A)$, satisfied by some nonrecursive r.e. set A, and such that $P(A)$ implies that A is incomplete?*

Marchenkov (3) proved that η-maximal semi-recursive r.e. sets are incomplete, and D. Miller later showed they are all low$_2$. Soare (4) (see ref. 5, p. 73) gave a property characterizing low r.e. sets, and Ambos-Spies and Nies (6) gave a property characterizing r.e. sets whose degrees are *cappable* (i.e. halves of a minimal pair). However, these three properties are all non $\mathcal{E}$-definable by Theorem 3 and Theorem 2′ below, respectively.

Post's Problem was solved by Friedberg (7) and independently Muchnik (8) with the introduction of the priority method. Post's Program remained open, however, and stimulated much research on the more general and fundamental question of studying the relationship between the algebraic structure of an r.e. set A and deg(A). In 1963 Sacks (9) (Q(3), p. 172) listed it as one of his five open problems on degrees. Shoenfield (10) and Sacks (11) and (9) introduced the infinite injury priority method to further classify r.e. degrees. Using this and other methods Martin (12) showed that the degrees of maximal r.e. sets (i.e. A such that A^* is a coatom in $\mathcal{E}^*$, the quotient lattice of $\mathcal{E}$ modulo the ideal of finite sets) are exactly the high r.e. degrees a (those such that $\mathbf{a}' = 0''$, where $0 = \deg(\emptyset)$, and $A' = K^A$). Lachlan (13) and Shoenfield (14) showed that the degrees of coinfinite r.e. sets with no maximal supersets are exactly the non low$_2$ r.e. degrees $\mathbf{a}$ (i.e. $\mathbf{a}'' > 0''$). Martin's result generalized an earlier result by Yates that maximal sets could be complete and therefore no "thinness" property in the style of Post on $\overline{A} = \omega - A$, could guarantee incompleteness.

Soare (15) developed a new method for generating automorphisms of $\mathcal{E}$ and used it to show that all maximal sets are automorphic, so that no definable property together with maximality ensures incompleteness. Cholak, Downey and Stob (16) used this automorphism method to prove that for every coinfinite r.e. A there is a complete r.e. B such that $\mathcal{L}(A) \cong \mathcal{L}(B)$, and hence no $\mathcal{E}$-definable property of $\overline{A}$ alone could answer Question 1 affirmatively. The automorphism method was widely used to produce new homogeneity properties for r.e. sets (see ref. 5, Chapters XV and XVI), and it was proposed that a negative solution to Post's Program could be obtained by positively settling the following.

Question 2 *For every nonrecursive r.e. set A does there exist an automorphism Φ of $\mathcal{E}$ such that $\Phi(A)$ is complete?*

This seemed quite promising and a number of positive partial results were obtained (see ref. 5, p. 379). However, in 1985 Harrington showed that the automorphism Φ of Question 2, even if it exists, must necessarily be non Δ^0_3, and so more complicated than the current techniques for building automorphisms would allow. This means that there is a *dynamic* obstacle (see §1) to a positive answer to Question 2. In 1984 Harrington (see ref. 5, p. 339) initiated the methodology for converting dynamic obstacles into definable properties. By pressing these ideas further Harrington and Soare now settle Question 1 positively (and therefore Question 2 negatively) in the following which is the main result of this paper.

Theorem 1 *There is a nonempty $\mathcal{E}$-definable property $Q(A)$ such that every r.e. set A satisfying $Q(A)$ is nonrecursive and incomplete. (Furthermore, all r.e. sets A satisfying $Q(A)$ are not only incomplete but also deg(A) forms half of a minimal pair of r.e. degrees.)*

Note that every r.e. set A satisfying Q (see §1) is a major subset and hence has high degree. A closely related property includes even sets of low degree. We say (in the style of (17)) that set B is *hemi-Q*, written $HQ(B)$, if there is an r.e. set A satisfying $Q(A)$ such that A can be split into the disjoint union of nonrecursive r.e. sets B and C. Note that if $HQ(B)$ then $B \leq_T A$ so B must also be incomplete. Since $HQ(B)$ is $\mathcal{E}$-definable, any automorphic image of B must also be incomplete. Hence, there is a nonrecursive low r.e. set B which cannot be mapped to a complete set by any automorphism of $\mathcal{E}$. In the direction of pressing progress on Question 2 as far as possible, Harrington and Soare proved a theorem almost complementary to Theorem 1.

Theorem 2 *If A is any r.e. set of promptly simple degree then A is effectively automorphic to a complete set, and this automorphism can be found uniformly (in an r.e. index for A and an index for the recursive function witnessing that A has promptly simple degree).*

See ref. 5 for a definition of promptly simple sets and degrees and for all other definitions and notation used here. (Theorem 2 strengthens a result by Downey, Cholak, and Stob (16) which used the much stronger hypothesis that A is a promptly simple set, and unlike Theorem 2, their proof is neither uniform nor effective.) Harrington and Soare have proved Theorem 2′, which asserts that Theorem 2 holds for sets A in a strictly larger class of r.e. degrees called *almost promptly simple* degrees, and hence the class of non promptly simple (i.e. *tardy*) degrees, $M = R - PS$ (i.e. the degrees of halves of minimal pairs) is not *invariant* as defined below. Harrington and Soare also believe that they can characterize exactly the class of degrees containing an r.e. set not automorphic to a complete set.

A major open problem has been to determine which subclasses of the r.e. degrees $\mathbf{R}$ (particularly which jump classes $\mathbf{H}_n$ and $\mathbf{L}_n$ and their complements) are *invariant.* A class $\mathbf{C}$ of r.e. degrees is *invariant* if it is the set of degrees of some class $\mathcal{C}$ of r.e. sets which is invariant under automorphisms of $\mathcal{E}$ (e.g. if $\mathcal{C}$ is $\mathcal{E}$-definable). For example, the result by Martin above shows that $\mathbf{H}_1$ is invariant, and those by Lachlan and Shoenfield above that $\overline{L}_2$ is invariant, where $\mathbf{H}_n = \left\{\mathbf{a} \in \mathbf{R} : \mathbf{a}^{(n)} = 0^{(n+1)}\right\}$, $\mathbf{L}_n = \left\{\mathbf{a} \in \mathbf{R} : \mathbf{a}^{(n)} = 0^{(n)}\right\}$, and $\overline{\mathbf{C}} = \mathbf{R} - \mathbf{C}$. For over 15 years these and the trivial classes $\mathbf{L}_0$ and $\mathbf{H}_0$ have been the only ones for which the answer to invariance was known. In the direction of Theorem 2 and Question 2, Harrington and Soare have proved:

Theorem 3 *For every nonrecursive r.e. set A there is an r.e. set B which is high (i.e. deg(B') = $0''$) such that A is Δ^0_3-automorphic to B.*

An immediate corollary is that for every $n > 0$, the classes $\mathbf{L}_n$ and $\overline{H}_n$ are noninvariant. This automorphism is not effective, however, but requires the new tree methodology invented in 1984 by Harrington for generating noneffective Δ^0_3-automorphisms of $\mathcal{E}$, which he used to show that for every r.e. set A, $\emptyset <_T A <_T \emptyset'$, there is an r.e. set B automorphic to A such that $B \not\leq_T A$ (see ref. 5, p. 379). In 1988–1990 Harrington, Lachlan, and Soare further developed this method to prove: for every low$_2$ coinfinite r.e. set A, $\mathcal{L}(A) \cong \mathcal{E}$; and $\overline{L}_1$ is not invariant. These are the first uses of automorphisms of $\mathcal{E}$ to prove noninvariance of jump classes of degrees.

1 The proof of Theorem 1

To state the crucial property $Q(A)$ we first need some definitions. **Warning:** From now on all sets $A, B, C, \ldots W, X, Y$ will be r.e. with or without explicit mention thereof. During a given point in the construction of an r.e. set X we let X denote the finite set of elements enumerated in X so far, and let X_s denote the approximation to X by the end of stage s.

Definition 1 *(i) A subset $A \subset C$ is a* major subset *of C(written $A \subset_m C$) if $C - A$ is infinite and for all e,*

$$\overline{C} \subseteq W_e \implies \overline{A} \subseteq^* W_e.$$

(Note that if $A \subset_m C$ then both A and C are nonrecursive.)

(ii) $A \sqsubset B$ if there exists C such that $A \sqcup C = B$ (i.e. $A \cup C = B$ and $A \cap C = \emptyset$).

(iii) If $\{X_s\}_{s\in\omega}$ and $\{Y_s\}_{s\in\omega}$ are recursive enumerations of r.e. sets X and Y define

$$X \setminus Y = \{z : (\exists s)[z \in X_s - Y_s]\},$$

the elements enumerated in X before Y and $X \searrow Y = (X \setminus Y) \cap Y$, the elements enumerated first in X and later in Y.

Properties of r.e. sets stated in terms of $X \setminus Y$ are $X \searrow Y$ are called *dynamic* properties because they refer to the *order* of enumeration of elements in X and Y, but $\mathcal{E}$-definable properties are *static* because they refer only to the elements eventually in X or Y. The known methods to generate automorphisms of $\mathcal{E}$ require meeting certain dynamic hypotheses (see ref. 5, p. 352). If we encounter an obstacle to meeting these dynamic hypotheses, we attempt to convert that obstacle into a static property like $Q(A)$ to show that the intended automorphism cannot exist.

Definition 2 $\quad Q(A) : (\exists C)_{A\subset_m C}\ (\forall B \subseteq C)(\exists D \subseteq C)(\forall S)_{S\sqsubset C}$ [

$$[B \cap (S - A) = D \cap (S - A)] \tag{1}$$

$$\implies (\exists T)[\overline{C} \subset T \ \&\ A \cap (S \cap T) = B \cap (S \cap T)]]. \tag{2}$$

This property $Q(A)$ (and the following proofs) should be visualized in the context of a two person game for r.e. sets in the sense of Lachlan (ref. 18) between the $\exists$-player (whom we call RED) who plays the r.e. sets A, C, D and T and the $\forall$-player (called BLUE) who plays the r.e. sets B and S. Intuitively, property $Q(A)$ asserts that if BLUE satisfies (1) (namely by making B copy D on $S - A$) then RED must construct $T \supset \overline{C}$ satisfying (2) (namely by making A copy B on $S \cap T$). Clearly $Q(A)$ implies $\emptyset <_T A$ because $A \subset_m C$. Hence, to establish Theorem 1 BLUE must first prove:

Lemma 1 $(\forall A)[Q(A) \implies K \not\leq_T A]$.

Proof of Lemma 1. Fix A and $C \in \mathcal{E}$ such that A satisfies $Q(A)$ via C, and fix enumerations of A and C such that $A \subseteq C \searrow A$. If $K \leq_T A$, then we can effectively list recursive functionals Ψ_i such that $W_i = \Psi_i^A$. If BLUE can uniformly (in e) enumerate a set P such that $P \neq \Psi_e^A$, then BLUE will have constructed a recursive function i such that $W_{i(e)} \neq \Psi_e^A$. An application of the recursion theorem then produces an e such that $W_{i(e)} = W_e \neq \Psi_e^A$ for the desired contradiction. Thus, to prove Lemma 1, BLUE must simply enumerate P to satisfy the requirement, $\mathcal{R} : P \neq \Psi^A$, where Ψ is a single recursive functional. Let $\psi(x)$ denote the use function corresponding to $\Psi(x)$.

To utilize the hypothesis $Q(A)$ BLUE will first split C into the disjoint union of uniformly r.e. sets $\{S_i\}_{i\in\omega}$, written $C = \sqcup_{i\in\omega} S_i$, and then on S_i BLUE will play B against $D = W_i$ to satisfy (1). Since $Q(A)$ holds, RED must reply with $T =$ some W_j to satisfy (2). Now BLUE will use a Π_2^0 guessing procedure (described in §1.2 below) to determine the correct values of i and j. However, to better explain the basic module for satisfying requirement $\mathcal{R}$ we will assume in §1.1 three simplifying hypotheses (discharged later in §1.2), the first of which asserts that BLUE has fixed the correct i and j so that BLUE is playing single sets B and S and has the indices i and j (respectively) of single r.e. sets D and T such that if BLUE satisfies (1) then RED satisfies (2).

1.1 The basic module for requirement $\mathcal{R}$ under simplifying assumptions

Now BLUE begins to satisfy (1) by first arranging that on $S - A$,

$$B \subseteq D \searrow B. \tag{3}$$

Hence, RED must ensure that on $S \cap T$,

$$A \subseteq B \searrow A, \tag{4}$$

because if $x \in (S \cap T \cap A) \setminus B$ then BLUE can restrain $x \in \overline{B}$ forever thereby refuting (2) while still maintaining (1) by ensuring (3) and (6) on $S - A$.

Now (3) and (4) together ensure that on $T \cap S$,

$$A \subseteq D \searrow B \searrow A. \tag{5}$$

To achieve the rest of (1) BLUE will occasionally enumerate x in B for every x currently in $(D - B) \cap (S - A)$, i.e. on $S - A$ BLUE will play

$$D - B = \emptyset. \tag{6}$$

This will force RED to ensure (2) by enumerating in A all x currently in $(B - A) \cap (S \cap T)$ so that on $S \cap T$

$$B - A = \emptyset. \tag{7}$$

As a second simplifying assumption BLUE assumes in §1.1 that if (2) holds for T then (2) also holds with T replaced by a certain set $U \subseteq T$ which will be played by BLUE and which also satisfies

$$U \cap C \subseteq^* S. \tag{8}$$

(BLUE will discharge this assumption in §1.2.)

But $A \subset_m C$ and $\overline{C} \subseteq U$ (from (2)) imply

$$\overline{A} \subset^* U, \tag{9}$$

so from (8) and (9) we get

$$(C - A) \subset^* S. \tag{10}$$

As a third and final simplifying assumption BLUE will assume that $\subset^*$ is replaced by $\subset$ in (8), (9), and (10) so that the action below to satisfy $\mathcal{R}$ is never injured.

The basic module for $\mathcal{R}$:

Step 1. Choose a fresh x not yet in P. Wait for a stage s such that $\Psi_s^A(x)\downarrow = 0$ with use $\psi_s(x) = u$ such that

$$(\forall y \leq u)[y \in A_s \ \vee \ y \in (U_s - C_s) \ \vee \ y \in (C_s - B_s) \cap (S_s \cap U_s)]. \tag{11}$$

(While waiting BLUE plays (3) and (6) to ensure (4) and (7). Hence, T and therefore U in place of T both satisfy (2) so $y \in \overline{A}$ satisfies $y \in U - C$ if

$y \in \overline{C}$, and if $y \in C - A$ then $y \in (C - B) \cap (S \cap U)$ by (7) – (10). Thus, if $\Psi^A(x)\downarrow = 0$ then (11) must eventually occur.)

Step 2. When (11) occurs for x then enumerate $x \in P$ and restrain all $y \leq \psi_s(x)$ from later entering B.

We claim that no $y \leq u = \psi_s(x)$, $y \notin A_s$, later enters A. (Hence, $\Psi^A(x) = 0$ but $P(x) = 1$, so requirement $\mathcal{R}$ is satisfied by x.) First suppose that $y \leq u$ satisfies the third clause of (11). Since y never enters B, y can never enter A because (4) holds on $S \cap U$ since $U \subseteq T$. Secondly, suppose that $y \leq u$ satisfies the second clause of (11) and y later enters A. Then y must first enter $C = S \sqcup \hat{S}$ (where $\hat{S}$ is the r.e. set $C - S$ guaranteed by $S \sqsubset C$ in (1)) at which time y immediately enters S (rather than $\hat{S}$) by (8) because $y \in U$ already. But now y satisfies the third clause of (11) and must remain in $\overline{A}$ as in the preceding paragraph. This proves the claim, and completes the description of the basic module.

In the pure basic module described above, the B-restraint imposed at Step 2 is never injured. In the full contruction in §1.2, this restraint may be injured by another requirement which causes RED to enumerate in A some $y \leq \psi_s(x)$. In this case BLUE modifies the basic module by dropping the former B-restraint, performing (6) for all y currently in $(D - B) \cap (S - A)$, and therefore restoring (1). Next BLUE chooses a fresh x, and restarts the basic module.

1.2 Discharging the three simplifying assumptions

Let $\{(D_i, T_j)\}_{i,j\in\omega}$ be an effective listing of all pairs of r.e. sets. Below BLUE will define r.e. sets $\{S_{i,j}\}_{i,j\in\omega}$ such that $C = \sqcup_{i,j\in\omega} S_{i,j}$. Now BLUE begins by playing for every i and j the set B on $S_{i,j}$ against D_i to satisfy (3) and (6) and therefore (1). Hence, (1) is also satisfied by the sets B, D_i, and $S_i = \sqcup_{j\in\omega} S_{i,j}$. Thus, for some j, T_j must satisfy (2) and hence (4) and (7) for B, D_i, and S_i, and therefore also for B, D_i, and $S_{i,j}$. Let $\alpha = \langle i, j\rangle$, the image under the standard pairing function, and let D_α, S_α, and T_α denote D_i, $S_{i,j}$, and T_j, respectively. (For every α, BLUE performs the modified basic module of the last paragraph of §1.1 on (B, D_α, S_α). If some α permanently imposes restraint then $\Psi(x)\downarrow \neq P(x)$ and we are done. If not then no α-restraint on B is permanent, so (1) is satisfied by BLUE .) For each α the conjunction of all the conditions in the matrices of (1) and (2) (with D, S, T replaced by

D_α, S_α, and T_α respectively) is a Π^0_2 condition $F(\alpha)$. Hence, there is an r.e. sequence of r.e. sets $\{Z_\alpha\}_{\alpha\in\omega}$ such that for every α, $F(\alpha)$ holds iff $|Z_\alpha| = \infty$. Define U_α by

$$x \in U_{\alpha,s} \Longleftrightarrow x \in U_{\alpha,s-1} \ \vee \ [x \in T_{\alpha,s} - C_s \ \&\ x \leq |Z_{\alpha,s}|]. \tag{12}$$

If $x \in C_{s+1} - C_s$ choose the least α such that $x \in U_{\alpha,s}$, and enumerate x in $S_{\alpha,s+1}$. (If no such α exists enumerate x in $S_{x,s+1}$.) Let α be the least β such that Z_β is infinite. Hence, D_α, S_α, T_α, and U_α satisfy the first two simplifying assumptions in §1.1 including (8), because by (12) Z_β and hence U_β and S_β are finite for every $\beta < \alpha$.

Hence, (8), (9), and (10) hold with $\subset^*$, and therefore hold with $\subset$ for all $z \geq$ some k. Thus, for each $\alpha, k \in \omega$ we have an $\langle\alpha, k\rangle$-strategy which behaves exactly like the basic module for α in §1.1 except that the $\langle\alpha, k\rangle$-strategy operates with the quantifier "$(\forall y \leq u)$" in line (11) replaced by "$(\forall y)_{k\leq y\leq u}$."

These strategies are arranged as in the usual tree method (see ref. 5, Chapter XIV) so that when the $\langle\alpha, k\rangle$-strategy acts under Step 2, it resets the $\langle\alpha', k'\rangle$-strategy for every $\alpha' > \alpha$, or $\alpha' = \alpha$ and $k' > k$, and prevents that strategy from later acting unless the Step 2 restraint imposed by the $\langle\alpha, k\rangle$-strategy is first injured.

Fix the least α satisfying $F(\alpha)$. For each k the conjunction of (8), (9), and (10) (with $\subset^*$ replaced by $\subset$ and measured with respect to all $z \geq k$) is a Π^0_2 condition $G(\alpha, k)$. The least k satisfying $G(\alpha, k)$ gives the "true path" node $\langle\alpha, k\rangle$ for which the $\langle\alpha, k\rangle$-strategy succeeds as in §1.1. (Notice that here unlike §1.1 the $\langle\alpha, k\rangle$-strategy may be injured finitely often if it has assigned B-restraint $r(s) = u$ and later some $y < k \leq u$ enters A. In this case the $\langle\alpha, k\rangle$-strategy begins anew on a fresh x, but such injury can occur at most k times.) This proves Lemma 1.

1.3 Proof of Lemma 2

To complete the proof of Theorem 1 we need to prove:

Lemma 2 $(\exists A)[Q(A)]$.

Proof. This proof is very similar to the standard proof (see ref. 5, p. 194) that every nonrecursive r.e. set C has a small major subset A ($A \subset_{sm} C$) to which the reader should now refer. Let C be any nonrecursive r.e. set. (If we choose C simple (maximal) then A will be simple (r-maximal).) To make $A \subset_m C$ it suffices to meet for every e the requirement,

$$\mathcal{P}_e: \quad \overline{C} \subseteq W_e \implies \overline{A} \subseteq^* W_e.$$

Replace W_e by $V_e = \bigcup_{s\in\omega} V_{e,s}$ defined by

$$x \in V_{e,s} \iff x \in V_{e,s-1} \ \vee \ [x \in (W_{e,s} - C_s) \ \& \ (\forall y \leq x)[y \in W_{e,s} \cup C_s]]. \quad (13)$$

Note that $C \searrow V_i = \emptyset$, for every i. Define the e-state $\sigma(e, x, s) = \{i : i \leq e \ \& \ x \in V_{i,s}\}$, with the usual ordering of e-states. Let $C = f(\omega)$ for f a 1:1 recursive function, and let $c_i = f(i)$. Let $C_s - A_s = \{d^s_0, d^s_1, \cdots\}$, in the ordering induced by $\{c_0, c_1, \cdots\}$. (Hence, if $x = d^s_i = d^t_j$ for $t > s$, then $j \leq i$.)

The strategy for $\mathcal{P}_e$ is as follows. If $i \geq e$, and $j > i$ is minimal such that $\sigma(e-1, d^s_i, s) = \sigma(e-1, d^s_j, s)$, $d^s_i \notin V_{e,s}$, and $d^s_j \in V_{e,s}$ then $\mathcal{P}_e$ *wants to enumerate* into A all the elements $\{d^s_k : i \leq k < j\}$ (but subject to the negative restraint by $\mathcal{N}_i$, $i \leq e$, as described below).

Let $\{B_i : i \in \omega\}$ and $\left\{(S_j, \hat{S}_j) : j \in \omega\right\}$ be an effective listing of all r.e. sets and all pairs of r.e. sets respectively. Let RED play D_i against B_i, $D_i \subseteq C$, and also construct $T_{i,j}$ to meet (2) if BLUE satisfies (1). Let $\alpha = \langle i, j\rangle$, and let D_α, B_α, S_α, $\hat{S}_\alpha$, and T_α denote D_i, B_i, S_j, $\hat{S}_j$, and $T_{i,j}$ respectively. For each α the conjunction of the matrix of (1) for $(B_\alpha, D_\alpha, S_\alpha)$ with the conditions $B_\alpha \subseteq C$, and $S_\alpha \sqcup \hat{S}_\alpha = C$ is a Π^0_2 relation $F(\alpha)$. Let $\{Z_\alpha\}_{\alpha\in\omega}$ be an r.e. array of r.e. sets such that $F(\alpha)$ holds iff $|Z_\alpha| = \infty$.

Define T_α by

$$x \in T_{\alpha,s} \iff x \in T_{\alpha,s-1} \ \vee \ [x \in \overline{C}_s \ \& \ x \leq |Z_{\alpha,s}|]. \quad (14)$$

The negative requirement $\mathcal{N}_\alpha$ on A asserts that if (1) holds for $(B_\alpha, D_\alpha, S_\alpha)$ then (2) holds for $(B_\alpha, S_\alpha, T_\alpha)$. The strategy for $\mathcal{N}_\alpha$ is this. If $x \in T_\alpha \searrow C$, then $\mathcal{N}_\alpha$ restrains x from A until $x \in S_\alpha \sqcup \hat{S}_\alpha$. (If the latter never occurs then $F(\alpha)$ fails so Z_α and T_α are finite, and only finitely many such $x \in T_\alpha \cap C$ are permanently restrained by $\mathcal{N}_\alpha$.) If $x \in \hat{S}_\alpha$ then $\mathcal{N}_\alpha$ imposes no further restraint on x.

If $x \in S_\alpha$, and while $x \in \overline{A}$, x is enumerated in $B_\alpha \setminus D_\alpha$ then $\mathcal{N}_\alpha$ restrains x from both D_α and A forever (unless some $\mathcal{P}_e$, $e < \alpha$, enumerates x in A).

(If $\mathcal{N}_\alpha$ successfully keeps $x \in \overline{D}_\alpha \cap \overline{A}$ then x violates (1) so $F(\alpha)$ fails and Z_α and T_α are finite.) Otherwise, suppose that (3) holds on $S_\alpha - A$.

If $x \in S_\alpha$ and some $\mathcal{P}_e$, $e \geq \alpha$, wants to enumerate x in A then $\mathcal{N}_\alpha$ first enumerates x in D_α and then restrains x from A until x is enumerated in B_α at which time $\mathcal{N}_\alpha$ *releases* x (forever). (If x remains in $\overline{B}_\alpha$ forever then $x \in (D_\alpha - B_\alpha) \cap (S_\alpha - A)$ so x violates (1) and again T_α is finite.) Hence, in any of the three cases $\mathcal{N}_\alpha$ permanently restrains at most finitely many elements.

We now combine the $\mathcal{P}_e$ and $\mathcal{N}_\alpha$-strategies to give the full construction of A. If $x \in C_s$, choose the least e (if any) such that $\mathcal{P}_e$ wants to enumerate x in A. We say that $\mathcal{P}_e$ *controls* x at s. If e controls x at s then before enumerating x in A, $\mathcal{P}_e$ performs the following steps during stages $t \geq s$. (Some of these steps may never terminate in which case $\mathcal{P}_e$ never enumerates x in A.) First $\mathcal{P}_e$ waits until $x \in S_\alpha \sqcup \hat{S}_\alpha$ for all $\alpha \leq e$. Next let $\alpha_0, \alpha_1, \cdots, \alpha_n$ be a listing of all $\alpha \leq e$ such that $x \in S_\alpha$ and $D_\beta \neq D_\alpha$ for all $\beta < \alpha$, with $x \in S_\beta$.

For each k, $0 \leq k \leq n$, we make x pass through the $\mathcal{N}_{\alpha_k}$-strategy above (also called the $\mathcal{N}_{\alpha_k}$*-gate*) in the order $\mathcal{N}_{\alpha_n}, \ldots, \mathcal{N}_{\alpha_1}, \mathcal{N}_{\alpha_0}$. Hence, for example, when x is released by the $\mathcal{N}_{\alpha_2}$-gate by being enumerated in B_{α_2}, RED then enumerates $x \in D_{\alpha_1}$ and waits for x to be released by $\mathcal{N}_{\alpha_1}$ by being enumerated in B_{α_1}. When x is released by the $\mathcal{N}_{\alpha_0}$-gate RED enumerates x in A.

Notice that if e controls x at s, it is possible for some different e' to control x at some $t > s$, but in this case $e' < e$, so the previous action by $\mathcal{P}_e$ before stage t is entirely compatible with the action of $\mathcal{P}_{e'}$ at stages $t' \geq t$.

Now for every e, requirement $\mathcal{P}_e$ is satisfied because every $\mathcal{N}_\alpha$ can permanently restrain only finitely many elements. To see that $\mathcal{N}_\alpha$ is satisfied, note that the definition of V_e in (13) ensures that either $\overline{C} \subset V_e$ or V_e is finite. Let G_α be the intersection of those V_e for $e < \alpha$ such that V_e is infinite. Then, modulo finite sets, $\mathcal{N}_\alpha$ is satisfied by $\tilde{T}_\alpha = T_\alpha \bigcap G_\alpha$, whose strategy guarantees that if (1) holds for $(B_\alpha, D_\alpha, S_\alpha)$ then (2) holds for $(B_\alpha, S_\alpha, \tilde{T}_\alpha)$. (Adjust $\tilde{T}$ for the finite set of elements permanently restrained by $\mathcal{N}_\beta$ or enumerated in A by $\mathcal{P}_e$ for $\beta < \alpha$ or $e < \alpha$ and V_e finite.) This completes the proof of Lemma 2 and hence of Theorem 1. ■

We can prove that if $Q(A)$ holds via C then $A \subset_{sm} C$. Because of the similarity of the proof of Lemma 2 to the small major subset construction, it is natural to ask whether the Q-like property $\hat{Q}(A) : (\exists C)[A \subset_{sm} C]$ guarantees

$A <_T K$. This is false, but $\hat{Q}(A)$ implies that A is not a promptly simple *set*.

Note that if $Q(A)$ holds then A does not have promptly simple degree (using the characterization in ref. 5, p. 284, line (1.6)), because for any candidate p for a prompt function for (1.6), BLUE defines an r.e. set W_e by enumerating u in W_e at s if (11) holds for u, and BLUE keeps all $y \leq u$ currently in $\overline{B}$ there (and hence in $\overline{A}$) until after stage $p(s)$.

The full proofs of Theorems 2 and 3 will appear elsewhere. We comment very briefly on these proofs now. We assume familiarity with the automorphism method of ref. 5, Chapter XV, whose notation and definitions we now use. To prove Theorem 2 we start to build an r.e. set B and a permutation p which induces an effective isomorphism from $\mathcal{L}^*(A)$ to $\mathcal{L}^*(B)$. Simultaneously, we use the hypothesis that A is of promptly simple degree to find for each e a full e-state ν_e such that $\nu_e \cap \overline{A} = \infty$, and $\nu_e \searrow A = \infty$, i.e. infinitely many elements reside permanently in $\overline{A}$ in state ν_e and infinitely many enter A while in state ν_e. We simultaneously build a Turing reduction Ψ such that $K = \Psi^B$, and ensure that for all $n \geq e$ the use function $\psi(n)$ is placed on an element of $\overline{B}$ in state ν_e. Hence, elements $\psi(n)$ later enumerated in B for coding if n enters K will be covered by elements in $\nu_e \searrow A$ so that the method of the Extension Theorem will produce an automorphism.

For Theorem 3 the coding strategy is similar but more subtle because we have only the weaker hypothesis that A is nonrecursive. Furthermore, the positive requirements $\mathcal{P}_n$ to make B high can now each enumerate an infinite recursive set into B. This forces the automorphism Φ mapping A to B to be noneffective because the definition of $\Phi(W_e)$ will depend upon the correct guess about the recursive set contributed to B by $\mathcal{P}_n$ for every $n < e$. This is handled by our new noneffective Δ^0_3-automorphism method which builds a tree of possible automorphisms. The "true path" through this tree produces the final automorphism Φ. (After reading a preliminary announcement of Theorem 3 in August, 1990, P. Cholak announced (private communication) that he had independently proved Theorem 3, using this new Δ^0_3-automorphism method which Harrington had previously described to him in outline form.)

Acknowledgements:

The first author was supported by National Science Foundation Grant DMS 89-10312, and the second author by National Science Foundation Grant DMS 88-07389. This research was mostly carried out while the authors were visiting the Mathematical Sciences Research Institute in Berkeley, California, during the Special Year in Mathematical Logic, from September 1, 1989 through August 24, 1990, partially supported by National Science Foundation Grant DMS 85-05550. The authors are grateful to C. G. Jockusch, Jr. and the referees for several suggestions and corrections on the first draft of this paper.

REFERENCES

1. Gödel, K. (1931), *Monatsh. Math. Phys.* **38**, 173–198.
2. Post, E. L. (1944) *Bull. Am. Math. Soc.* **50**, 284–316.
3. Marchenkov, S. S. (1976) *Mat. Zametki* **20**, 473–478.
4. Soare, R. I. (1977) *J. Symbolic Logic* **42**, 545–563.
5. Soare, R. I. (1987) *Recursively Enumerable Sets and Degrees: A Study of Computable Functions and Computably Generated Sets,* (Springer-Verlag, Heidelberg).
6. Ambos-Spies, K., and A. Nies, Cappable recursively enumerable degrees and Post's program, *Archive of Math. Logic*, in press.
7. Friedberg, R. M. (1957) *Proc. Natl. Acad. Sci. USA* **43**, 236–238.
8. Muchnik, A. A. (1956) *Dokl. Akad. Nauk SSSR, N.S.* **108**, 194–197.
9. Sacks, G. E. (1963) *Degrees of Unsolvability*, Annals of Mathematics Studies, no. 55 (Princeton Univ. Press, Princeton, NJ).
10. Shoenfield, J. R. (1961) *Fundam. Math.* **49**, 171–179.
11. Sacks, G. E. (1963) *Trans. Am. Math. Soc.* **108**, 223–239.
12. Martin, D. A. (1966) *Z. Math. Logik Grundlag. Math.* **12**, 295–310.
13. Lachlan, A. H. (1968) *J. Symbolic Logic* **33**, 431–443.
14. Shoenfield, J. R. (1976) *J. Symbolic Logic* **41**, 695–696.
15. Soare, R. I. (1974) *Ann. Math. (2)* **100**, 80–120.
16. Cholak, P., Downey, R., & Stob, M., Automorphisms of the lattice of recursively enumerable sets: promptly simple sets, *Trans. Am. Math. Soc.*, in press.
17. Downey, R., & Stob, M., Automorphisms of the lattice of recursively enumerable sets: orbits, *Advances in Math.*, in press.
18. Lachlan, A. H. (1970) *Ann. Math. (2)* **91**, 291–310.